

# COVERING MONOPOLE MAP AND HIGHER DEGREE IN NON COMMUTATIVE GEOMETRY

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ABSTRACT. In this paper we develop analysis of the monopole maps over universal covering spaces of compact four manifolds. We induce a property on local properness of the covering monopole maps under the condition of closeness of the AHS complexes. In particular we construct a higher degree of the covering monopole maps when their linearized equations are isomorphic, which induces a homomorphism between K group of the group  $C^*$  algebras. It involves non linear analysis on the covering spaces, which is related to  $L^p$  cohomology. We also obtain various Sobolev estimates on the covering spaces.

As a possible application, we propose an aspherical version of  $\frac{10}{8}$  inequality, combining with Singer conjecture on  $L^2$  cohomology. It is satisfied for a large class of four manifolds which includes some complex surfaces of general type.

## 1. INTRODUCTION

In this paper we study analysis of the monopole maps over the universal covering spaces of compact four manifolds. Based on the idea of finite dimensional approximation, Bauer-Furuta constructed degree of the monopole maps, which recovers the gauge theoretic invariants in Seiberg-Witten theory [BF]. This aims at construction of a covering version of their construction.

Gauge theory gave a serious development of study of smooth structure in four dimension. In a general mechanism, it constructs moduli spaces which are given by solutions to some non linear elliptic equations modulo gauge symmetry, and their tangent spaces are given by the index bundles of the families of elliptic operators parametrized by the moduli spaces. So the Atiyah-Singer index theorem is the basic object of the local model of the moduli spaces.

Classical surgery theory tells us that fundamental group gives serious effects on smooth structure on manifolds. In high dimension, they are reduced to algebraic topology of group ring, which promoted study of geometry and analysis over their universal covering spaces. The Atiyah-Singer index theorem has been extensively developed over non compact manifolds so far. In particular the construction by Gromov and Lawson is fundamental and revealed a deep relation to the existence of positive scalar curvature metrics [GL]. Non commutative

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geometry created a new framework in geometry, which unifies analysis of the index theory over non compact spaces with surgery theory passing through representation theory [C].

Smooth structure is a core background in both fields of non commutative geometry and gauge theory. Actually based on the Atiyah-Singer index theorem, both theories gave serious development in differential topology. It would be quite natural to try to combine both theories by introducing a systematic tool to analyse smooth structure of four manifolds from the view point of their fundamental groups, and to construct moduli theory over non compact spaces in non linear non commutative geometry. This paper is the first step to attack this project by use of Seiberg-Witten theory and Bauer-Furuta invariants, in order to construct an infinite dimensional degree theory in non commutative geometry. This would also motivate to develop analysis of  $L^p$  cohomology theory which appears naturally in our subject, since it involves non linear analysis over non compact spaces.

Let us explain finite dimensional version of our construction. Let  $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a proper map. It induces a map between  $K$ - theory  $\varphi_* : K_*(\mathbb{R}^n) \rightarrow K_*(\mathbb{R}^n)$  by composition of functions  $f \in C_0(\mathbb{R}^n)$  to  $f \circ \varphi \in C_0(\mathbb{R}^n)$ . If a discrete group acts on  $\mathbb{R}^n$  and  $\varphi$  is  $\Gamma$ -equivariant, then it induces the map between the equivariant  $K$  theory:

$$\varphi_* : K_*^\Gamma(\mathbb{R}^n) \equiv K_*(\mathbb{R}^n \rtimes \Gamma) \rightarrow K_*(\mathbb{R}^n \rtimes \Gamma)$$

If the action of  $\Gamma$  on  $\mathbb{R}^n$  is free, then we have the map  $\varphi_* : K_*(\mathbb{R}^n/\Gamma) \rightarrow K_*(\mathbb{R}^n/\Gamma)$  over the classifying space, which is determined by the induced group homomorphism  $\varphi_* : \Gamma = \pi_1(\mathbb{R}^n/\Gamma) \rightarrow \Gamma$ .

One might say that our construction is its infinite dimensional version. Let us start from a non linear elliptic map between vector bundles, and extend it to the map  $F : L_k^2(E) \rightarrow L_{k-1}^2(F)$  between their Sobolev spaces. The straightforward analogue of degree in finite dimension does not exist over Hilbert space, since infinite dimensional unitary group is contractible. Higson, Kasparov and Trout introduced an infinite dimensional version of Bott periodicity by constructing some  $C^*$ -algebras of infinite dimensional Clifford algebras [HKT]. It allows us to induce Bott periodicity between Hilbert spaces in  $K$  theory. In this paper we combine the constructions of the Bauer-Furuta degree theory with Higson-Kasparov-Trout Bott periodicity, and introduce  $K$ -theoretic degree of the covering monopole map.

Our main aim here is to construct a covering monopole operator which is given by  $*$ -homomorphism between some  $C^*$ -algebras. In order to perform it, there are some analytic conditions which we have to

assume at present. One of them is closedness of the linearized operators. Such analysis has been developed extensively in relation with  $L^2$  cohomology theory, and we can find plenty of instances of four manifolds whose covering spaces satisfy such property. In this paper we construct the covering monopole maps between the Clifford algebras of the Hilbert spaces when the linearized maps are isomorphic. We also present examples of spaces which satisfy such property. General cases will be considered in later papers. We shall also include some basic analysis of the covering monopole maps over general four manifolds. In fact we will not assume isomorphisms of the linearized maps until section 6.

Let us recall construction of the Seiberg-Witten moduli space. Let  $M$  be an oriented compact four manifold, and  $S^\pm$  and  $L$  be the Hermitian rank 2 bundles and their determinant bundle associated to a  $\text{spin}^c$  structure. The Clifford multiplication  $T^*M \times S^\pm \rightarrow S^\mp$  defines a linear map  $\rho : \Lambda^2 \rightarrow \text{End}_{\mathbb{C}}(S^+)$  whose kernel is the sub-bundle of anti self dual 2 forms and the image is the sub-bundle of trace free skew Hermitian endomorphisms.

The space of configuration of the Seiberg-Witten map consists of the set of  $u(1)$  connections over  $L$  and sections of positive spinors. The map associates as:

$$F(A, \phi) = (D_A(\phi), F^+(A) - \sigma(\phi))$$

where  $A$  lifts to the connection over the spinors, and  $\sigma(\phi)$  is the trace free endomorphism:

$$(-i)(\phi \otimes \phi^* - 1/2|\phi|^2 \text{id})$$

considered as a self dual 2 form on  $M$  via  $\rho$ .

The gauge group acts on the space of configuration, which is the set of automorphisms of the principal  $\text{spin}^c$  bundle which cover the identity on the frame bundle. It is given by a map from  $M$  to the center  $S^1$  of  $\text{Spin}^c(4)$ .

The Seiberg-Witten map  $F$  is equivariant with respect to  $u(1)$  gauge group actions  $\mathfrak{G}$ , and its moduli space is given by all the set of solutions divided by gauge group actions:

$$\mathfrak{M} = \{(A, \phi) : F(A, \phi) = 0\} / \mathfrak{G}.$$

Now recall a basic differential topology. Let  $M$  and  $N$  be compact oriented manifolds of dimension  $n$ , and consider a smooth map  $f : M \rightarrow N$ . There are two different ways to extract degree of  $f$ , where one is to count the number of the inverse of a generic point of  $f$ . The other is to find a multiplication number of the pull-back  $f^* : H^n(N :$

$\mathbb{Z}) \rightarrow H^n(M : \mathbb{Z})$ . Let us consider the case when the dimension of the Seiberg-Witten moduli space has generic zero dimension. We may try to follow such two different interpretation of the degree, and try to construct degree of the Seiberg-Witten map by algebro-topological method. This is the basic idea of Bauer-Furuta theory.

Of course one of the big difference from finite dimensional case is that the spaces are Sobolev spaces which are non locally compact. Hence one needs more functional analytic ideas to perform such process. Let us recall its construction partly, which is based on a rather abstract formalism of homotopy theory over infinite dimensional spaces by A. Schwartz [S]. Let  $F = l + c : H' \rightarrow H$  be a Fredholm map so that the linearized map  $l$  is Fredholm and its non linear part  $c$  is compact. Then for  $r \gg 1$  and  $r$ -ball  $D_r \subset H'$ , the restriction of  $F$  on ‘large’ finite dimensional linear subspaces  $V' \subset H'$  turns out to be ‘asymptotically proper’ in some sense:

$$\text{pr} \circ F : V' \cap D_r \rightarrow V = l(V')$$

after projection to the image by  $l$ . Bauer-Furuta theory applies the framework to the monopole map. Seiberg-Witten map itself is not proper, and we modify it as follows. The monopole map  $\mu$  is defined for quadruplet  $(A, \phi, a, f)$  where  $A$  is a  $\text{spin}^c$  connection,  $\phi$  is a positive spinor (section of  $S^+$ ), and  $a$  and  $f$  are one form and locally constant function respectively. Then

$$\begin{aligned} \mu : \text{Conn} \times (\Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M)) &\rightarrow \\ \text{Conn} \times (\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)) & \\ (A, \phi, a, f) &\rightarrow (A, D_{A+a}\phi, F_{A+a}^+ - \sigma(\phi), d^*(a) + f, a_{\text{harm}}) \end{aligned}$$

where  $\mu$  is equivariant with respect to the gauge group action  $\mathfrak{G} = \text{map}(M, \mathbf{T})$ . We denote by  $\mathfrak{G}_0$  as the based gauge group.

The subspace  $A + \ker(d) \subset \text{Conn}$  is invariant under the free action of the based gauge group. Its quotient is isomorphic to the space of equivalent classes of flat connections  $\text{Pic}(M) = H^1(M : \mathbb{R})/H^1(M : \mathbb{Z})$ . Now we denote the quotient:

$$\begin{aligned} \mathfrak{A} &\equiv (A + \ker(d)) \times (\Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M))/\mathfrak{G}_0, \\ \mathfrak{C} &\equiv (A + \ker(d)) \times (\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M))/\mathfrak{G}_0. \end{aligned}$$

Then the monopole map descends to the fibered map over  $\text{Pic}(M)$ :

$$\mu : \mathfrak{A} \rightarrow \mathfrak{C}.$$

The extra factor  $H^1(M)$  will affect to present the appropriate indices of the linearized map. It will be overcome by choosing trivializations. As a general fact a Hilbert bundle over a compact space admits trivialization

so that the isomorphisms  $\mathfrak{A} \cong H' \times \text{Pic}(M)$  and  $\mathfrak{C} \cong H \times \text{Pic}(M)$  hold. Let us consider the composition with the projection:

$$\text{pr} \circ \mu : \mathfrak{A} \rightarrow H.$$

**Lemma 1.1** (BF). *Let  $M$  be a compact oriented smooth four manifold. The monopole map over  $M$  defines an element in the stable co-homotopy group, and the image of the degree map from the group coincides with the Seiberg-Witten invariant.*

In this paper we shall construct the Clifford  $C^*$ -algebras, and induce the following:

**Proposition 1.2.** *Suppose  $H^1(M; \mathbb{R}) = 0$ . The monopole map induces a  $*$ -homomorphism*

$$\mu : C^*(\mathfrak{A}) \rightarrow C^*(\mathfrak{C})$$

*between the Clifford algebras of Hilbert spaces such that the induced homomorphism on  $K$ -theory induces a map:*

$$\mu_* : K(C^*(\mathfrak{A})) \cong \mathbb{Z} \rightarrow \mathbb{Z} \cong K(C^*(\mathfrak{C})) \quad \mu_*(n) = \alpha n$$

*where  $\alpha \in \mathbb{Z}$  is the degree 0 Seiberg-Witten invariant.*

Our aim is to extend the construction of the  $*$ -homomorphism above over the universal covering spaces of compact oriented smooth four manifolds equivariantly with respect to the fundamental group actions.

**1.1. Main results and conjectures.** Let  $M$  be a compact oriented smooth Riemannian four manifold, and  $X = \tilde{M}$  be its universal covering space equipped with the lift of the metric. Let us fix a  $\text{spin}^c$  structure on  $M$ , and choose a solution  $(A_0, \psi_0)$  to the Seiberg-Witten equation over  $M$ . We denote the lift of  $A_0$  by  $\tilde{A}_0$  over  $X$ .

In this paper we shall introduce the covering monopole map at the base  $(A_0, \psi_0)$ :

$$\begin{aligned} \tilde{\mu} : L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow L_{k-1}^2(X; \tilde{S}^- \oplus (\Lambda_+^2 \oplus \Lambda^0) \otimes i\mathbb{R}) \oplus H^1(X) \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), d^*(a), [a]) \end{aligned}$$

where  $[a]$  is the orthogonal projection to the first  $l^2$  cohomology group.

If the AHS complex has closed range over  $X$ , then  $L^2$  cohomology groups  $H^*(X; \mathbb{R})$  are uniquely defined for  $* = 1, 2$ . Throughout this paper, we will assume closeness of the AHS complex over  $X$ .

Concerning the gauge group action, we will verify that the covering monopole map admits the global slice:

$$\begin{aligned} \tilde{\mu} : L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* &\rightarrow L_{k-1}^2(X; \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), [a]) \end{aligned}$$

which is  $\Gamma$  equivariant map, when the AHS complex has closed range.

The linearized operator of the covering monopole map over  $X$  is  $\Gamma$ -Fredholm whose  $\Gamma$ -index coincides with:

$$\begin{aligned} \dim_{\Gamma} d\tilde{\mu} &= \text{ind } D - \chi_{AHS}(M) - \dim_{\Gamma} H^1(X) \\ &= \text{ind } D - \dim_{\Gamma} H_+^2(X). \end{aligned}$$

where  $\chi_{AHS}(M) = b_0(M) - b_1(M) + b_2^+(M)$ .

**Remark 1.3.** Let us denote  $\text{Ker } d \subset L_{k+1}^2(X : \Lambda^1 \otimes i\mathbb{R})$ , and put  $\mathbf{A}_0 = \tilde{A}_0 + \text{Ker } d$ . A covering version of the Bauer-Furuta formalism is  $\mathfrak{G}_{k+1}(L) \rtimes \Gamma$  equivariant monopole map:

$$\begin{aligned} \tilde{\mu} : \mathbf{A}_0 \times L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow \\ \mathbf{A}_0 \times (L_{k-1}^2(X; \tilde{S}^- \oplus (\Lambda^0 \oplus \Lambda_+^2) \otimes i\mathbb{R}) \oplus H^1(X)) & \end{aligned}$$

The quotient space by the gauge group is fibered over the first  $L^2$  cohomology group:

$$\mathbf{A}_0 \times_{\mathfrak{G}_{k+1}(L)} L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cong H^1(X) \times L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R})$$

and the latter space is similar. By projecting to the fiber, we obtain  $\mathfrak{G}_{k+1}(L) \rtimes \Gamma$  equivariant monopole map:

$$\begin{aligned} \tilde{\mu} : H^1(X) \times L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow \\ L_{k-1}^2(X; \tilde{S}^- \oplus (\Lambda^0 \oplus \Lambda_+^2) \otimes i\mathbb{R}) \oplus H^1(X) & \end{aligned}$$

$\Gamma$ -index of the liberalized map is given by:

$$\dim_{\Gamma} d\tilde{\mu} = \text{ind } D - \chi_{AHS}(M)$$

which is a topological invariant of the base manifold  $M$ . However we encounter difficulty to analyze this space that it is never proper.

Let  $F = l + c : H' \rightarrow H$  be a smooth map between Hilbert spaces, where  $l$  is its linear part. For a finite dimensional linear subspace  $W \subset H$ , let us put  $W' = l^{-1}(W) \subset H'$ . Consider the restriction composed with the projection to  $W$ :

$$\text{pr} \circ F : W' \rightarrow W$$

Suppose  $W'$  is also finite dimensional and the restriction is proper. Then we can obtain the induced homomorphism:

$$(\text{pr} \circ F)^* : C_0(W) \rightarrow C_0(W').$$

Let us regard that this approximates the original map  $F : H' \rightarrow H$ , and try to construct ‘induced map’ from function spaces over  $H$  to the one over  $H'$ . When  $F$  is Fredholm, it can be reduced to restriction over a large but finite dimensional subspace, as was verified by Schwartz.

In our case of the covering monopole map, we have to construct an induced map between function spaces over infinite dimensional linear spaces. Our idea is to use the infinite dimensional Clifford algebras construction by [HKT], where the Clifford  $C^*$  algebras of Hilbert space  $S\mathfrak{C}(H)$  is constructed, which is given passing through a kind of limit of  $C_0(W, Cl(W))$  over all finite dimensional linear subspaces  $W \subset H$ .

Let  $H' = L_k^2(X; E)$  be realized by a Sobolev space for some vector bundle  $E$  over  $X$ , and  $D_r \subset H'$  be  $r$ -ball.

**Definition 1.1.** *Let  $F : H' \rightarrow H$  be a smooth map between Hilbert spaces.*

*It is strongly proper, if (1) the pre image of a bounded set is contained in some bounded set, and (2) the restriction of  $F$  on  $D_r \cap L_k^2(K; E)_0$  is proper, where they are any bounded set of the Sobolev spaces whose supports lie in compact subset  $K \subset\subset X$ .*

*$F$  is locally strongly proper, if it is strongly proper over the restriction on  $L_k^2(K; E)_0$  for any compact subset  $K \subset\subset X$ .*

In this paper we shall construct the following:

**Theorem 1.4.** *Suppose  $F$  is locally strongly proper, and  $l : H' \cong H$  gives a linear isomorphism.*

*Then it induces a  $\Gamma$ -equivariant  $*$ -homomorphism:*

$$F^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}(H').$$

*In particular it induces  $K$ -theoretic higher degree given by an element in the equivariant  $E$  theory:*

$$[F^*] \in E_\Gamma(\mathbb{C}, \mathbb{C}).$$

**Remark 1.5.**  *$F^*$  induces the homomorphism between  $K(S\mathfrak{C}(H))$  which is isomorphic to  $K(C^*(\Gamma))$  by [HKT] (see section 6). Combination with these induces a homomorphism on  $K$ -theory:*

$$F^* : K(C^*(\Gamma)) \rightarrow K(C^*(\Gamma)).$$

*In fact there are natural homomorphisms:*

$$E_\Gamma(\mathbb{C}, \mathbb{C}) \rightarrow KK(C^*(\Gamma), C^*(\Gamma)) \rightarrow Hom(K(C^*(\Gamma)), K(C^*(\Gamma)))$$

*where the second one is given by the intersection product:*

$$KK(\mathbb{C}, C^*(\Gamma)) \times KK(C^*(\Gamma), C^*(\Gamma)) \rightarrow KK(\mathbb{C}, C^*(\Gamma))$$

*under the identification with  $K(C^*(\Gamma)) = KK(\mathbb{C}, C^*(\Gamma))$ .*

*The image of  $[F^*]$  coincides with the homomorphism on  $K$  theory.*

In this paper we verify the following:

**Theorem 1.6.** *Suppose the AHS complex has closed range over  $L^2$  Sobolev spaces. Then the monopole covering map is locally strongly proper.*

In fact we verify a stronger property so that the restriction of the slice:

$$\begin{aligned} \tilde{\mu} : L_k^2(K; \tilde{S}^+)_0 \oplus L_k^2(X; \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* \\ \rightarrow L_{k-1}^2(X; \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) \end{aligned}$$

is strongly proper for each compact subset  $K \subset\subset X$ .

**Corollary 1.7.** *Suppose the AHS complex has closed range over  $L^2$  Sobolev spaces. Assume the following conditions:*

- (1) *The Dirac operator over the covering space is invertible, and*
- (2) *the second  $L^2$  cohomology of the AHS complex vanishes.*

*Then the covering monopole map gives a  $\Gamma$ -equivariant  $*$ -homomorphism:*

$$\tilde{\mu}^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}(H').$$

*In particular it induces the homomorphism on  $K$ -theory:*

$$\tilde{\mu}^* : K(C^*(\Gamma)) \rightarrow K(C^*(\Gamma)).$$

In the above case the  $\Gamma$ -dimension of the covering monopole map is equal to zero, since it coincides with the index of the Dirac operator over  $M$ . We shall present some examples of four manifolds whose covering spaces satisfy these conditions with respect to their spin structure.

**Remark 1.8.** *Asymptotic morphism is a notion between  $C^*$  algebras which is weaker than the usual  $*$ -homomorphism [CH], but still it induces a homomorphism between  $K$ -theory. We expect that one can have a way to construct a  $\Gamma$ -equivariant asymptotic morphism  $\tilde{\mu}^*$  from  $S\mathfrak{C}(H)$  to  $S\mathfrak{C}(H')$  over any covering monopole map.*

*So far we are done under the condition that the linearized operator gives an isomorphism. In general there appear non zero kernel and co-kernel subspaces which are both infinite dimensional if fundamental group is infinite. In order to perform this, we will have to use some method which stabilize these infinite dimensional spaces. We may have to use Kasparov's  $KK$  theory for general construction (see remark 1.3).*

**1.1.1. A new phenomena.** Suppose the AHS complex is closed, and consider the covering monopole map

$$\tilde{\mu} : L_k^2(X; \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \rightarrow L_{k-1}^2(X; \tilde{S}^- \oplus (\Lambda^0 \oplus \Lambda_+^2) \otimes i\mathbb{R}) \oplus H^1(X)$$

whose quotient map by  $L_{k+1}^2$  gauge group  $\mathfrak{G}$  over  $X$  is denoted by  $\tilde{\mu} : H' \rightarrow H$  in short.



Recall that we have chosen a solution  $(A_0, \psi_0)$  to the Seiberg-Witten equation over the base compact four manifold  $M$ . Let  $\tilde{\mathfrak{M}}(M)$  be all the set of such solutions over  $M$ . The moduli space  $\mathfrak{M}(M)$  is given as the quotient space by the gauge group  $\mathfrak{G}(M)$ . So the covering monopole map is parametrized by  $\tilde{\mathfrak{M}}(M)$  and hence give a family of the covering monopole maps:

$$\tilde{\mathfrak{M}}(M) \times H' \rightarrow \tilde{\mathfrak{M}}(M) \times H$$

$\mathfrak{G}(M)$  acts on both  $H'$  and  $H$ , since  $\mathfrak{G}(M) \cap \mathfrak{G}$  consists only of identity, and their actions of course commute.

Let us consider the quotient space:

$$\tilde{\mu} : (\tilde{\mathfrak{M}}(M) \times H')/\mathfrak{G}(M) \rightarrow (\tilde{\mathfrak{M}}(M) \times H)/\mathfrak{G}(M)$$

which is fibered over the moduli space  $\mathfrak{M}(M)$  on  $M$ . If we trivialize as

$$(\tilde{\mathfrak{M}}(M) \times H)/\mathfrak{G}(M) \cong \mathfrak{M}(M) \times H$$

and compose it with the projection, then we obtain the family of the covering monopole map:

$$\tilde{\mu} : \mathfrak{M}(M) \times H' \rightarrow H$$

The linearized map is  $\Gamma$ -Fredholm whose  $\Gamma$ -index is given by:

$$\dim_{\Gamma} d\tilde{\mu} = -\dim_{\Gamma} H^1(X)$$

Notice that the formula always hold even when the dimension of  $\mathfrak{M}(M)$  can take arbitrary.

**1.2. Relation with Baum-Connes theory.** It has been known that smooth structure on manifolds has deep connection with their fundamental groups. As far as four manifold theory, smooth structure has not yet much developed so far from the fundamental group view point, unlike to the high dimensional case. Here we would like to present a proposal on a group theoretic property of the Seiberg-Witten invariant as an application of Baum-Connes theory in non commutative geometry [C].

Roughly speaking Baum-Connes theory claims that  $K$  theories of the group  $C^*$  algebras  $C^*(\Gamma)$  and of the classifying spaces  $B\Gamma$  possess ‘same size’ of information. This allows us to compute the former group in terms of topological data of  $\Gamma$ . Let us recall the Baum-Connes map:

$$A : K_*(B\Gamma) \rightarrow K_*(C^*\Gamma)$$

and give its description roughly. For simplicity suppose  $B\Gamma$  is realized by a compact smooth manifold  $M$ . An element in  $K_*(B\Gamma)$  is given by a pair of  $\mathbb{Z}_2$  complex vector bundle  $E$  with a  $\text{spin}^c$  structure on  $M$ . They give the Dirac operator  $D_E$  over  $M$ , and all can be twisted by

the flat  $C^*(\Gamma)$  bundle  $\tilde{M} \times_{\Gamma} C^*(\Gamma) = E\Gamma \times_{\Gamma} C^*(\Gamma)$ . Both the kernel and cokernel spaces of the twisted Dirac operator admits structure of finitely generated projective  $C^*(\Gamma)$  modules. Its index as their formal difference gives an element in  $K_*(C^*\Gamma)$ .

Baum-Connes conjecture claims an isomorphism of  $A$  (more precisely the right hand side should be the reduced  $C^*$  algebras, but it always passes through  $K$  group of the full  $C^*$  algebras). It has been known to be true for many of classes of discrete groups, while it has not yet verified in full generality.

**Lemma 1.9.** *Assume the conclusion of corollary 1.6. Moreover suppose  $S\mathfrak{C}(H) \rtimes \Gamma$  admits a unique trace.*

*Then there is a real number  $\alpha \in \mathbb{R}$  so that the following diagram commutes:*

$$\begin{array}{ccc} K(C^*\Gamma) & \xrightarrow{\tilde{\mu}_*} & K(C^*\Gamma) \\ \downarrow \text{tr} & & \downarrow \text{tr} \\ \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R} \end{array}$$

*Proof.*  $\tilde{\mu}_*$  is induced from  $*$ -homomorphism between  $S\mathfrak{C}(H) \rtimes \Gamma$ . Let us compose it with the trace:

$$\text{tr} \circ \tilde{\mu} : S\mathfrak{C}(H) \rtimes \Gamma \rightarrow \mathbb{R}$$

It satisfies commutativity:

$$\begin{aligned} \text{tr}(\tilde{\mu}(ab)) &= \text{tr}(\tilde{\mu}(a)\tilde{\mu}(b)) \\ &= \text{tr}(\tilde{\mu}(b)\tilde{\mu}(a)) = \text{tr}(\tilde{\mu}(ba)). \end{aligned}$$

It follows from the uniqueness of the trace on  $S\mathfrak{C}(H) \rtimes \Gamma$  that there is a real number  $\alpha \in \mathbb{R}$  so that  $\text{tr} \circ \tilde{\mu}(a) = \alpha \text{tr}(a)$  hold for all  $a \in S\mathfrak{C}(H) \rtimes \Gamma$ .  $\square$

**Remark 1.10.** *We will call  $\alpha$  as the covering monopole invariant. It may happen that this invariant  $\alpha$  coincides with the Seiberg-Witten invariant on some class of four manifolds. To see it, one may follow some non linear version of the proof of the Atiyah's  $\Gamma$  index theorem.*

Let us consider some properties of the covering monopole invariant.

**Conjecture 1.11.** *Suppose  $M$  is aspherical with  $\pi_1(M) = \Gamma$ . Then:*

$$|\alpha| \in \text{ind } \Gamma = \{n \in \mathbb{Z} : \exists \Gamma_0 \subset \Gamma \text{ with } [\Gamma : \Gamma_0] = n\} \subset \mathbb{N}$$

*where  $\Gamma_0$  are subgroups of  $\Gamma$  and we put  $[\Gamma : \Gamma_0] = 0$  if  $\Gamma_0 = 1$ .*

*In particular  $\alpha$  is an integer.*

It follows from the Atiyah's  $\Gamma$  index theorem that the combination:

$$\text{tr} \circ A : K_*(B\Gamma) \rightarrow K_*(C^*\Gamma) \rightarrow \mathbb{R}$$

coincides with the usual index of  $D_E$ . So  $\alpha$  is actually integer under the BC conjecture.

*A possible approach:* Suppose  $X = B\Gamma$  is realized by an oriented closed manifold. Moreover assume that BC conjecture holds for  $\Gamma$ . Then  $|\alpha| = k \in \text{ind } \Gamma$  holds, if a continuous map  $f : B\Gamma \rightarrow B\Gamma$  exists so that the induced homomorphism  $f_* : K(B\Gamma) \rightarrow K(B\Gamma)$  gives the commutative diagram:

$$\begin{array}{ccc} K(C^*\Gamma) & \xrightarrow{\tilde{\mu}_*} & K(C^*\Gamma) \\ \uparrow A & & \uparrow A \\ K(B\Gamma) & \xrightarrow{f_*} & K(B\Gamma) \end{array}$$

Suppose  $f : B\Gamma \rightarrow B\Gamma$  is surjective. Then we claim that the induced homomorphism:

$$f_* : \Gamma \rightarrow \Gamma$$

satisfies the equality:

$$k = [\Gamma : f_*(\Gamma)] \in \text{ind } \Gamma$$

Actually consider the lift  $\tilde{f} : E\Gamma \rightarrow E\Gamma$ . There is  $g_1, \dots, g_k \in \Gamma$  such that for any fundamental domain  $D \subset E\Gamma$ , the image coincides with:

$$f(D) = \cup_i g_i(D).$$

The claim follows from the fundamental theorem on the covering space.

**1.3. Higher  $\frac{10}{8}$  conjecture.** Let us recall the following:

**Theorem 1.12 (F).** *Let  $M$  be a compact smooth spin four manifold. Then the inequality holds:*

$$b^2(M) \geq \frac{10}{8} |\sigma(M)|$$

where  $\sigma$  is the signature.

The original conjecture is a stronger version which bounds by  $\frac{11}{8} |\sigma(M)|$  and is still open. Furuta produced a stronger inequality, which bounds by  $\frac{10}{8} |\sigma(M)| + 2$ .

The proof uses a kind of finite dimensional approximation described here, with representation theoretic observation over the  $\mathbb{H}$  and  $\mathbb{R}$  which are both  $\text{Pin}_2$  modules.

We would like propose a higher version of  $\frac{10}{8}$  inequality. The following argument essentially uses the covering version of the inequality.

**Conjecture 1.13.** *Suppose  $M$  is a compact aspherical smooth four manifold with even type intersection form. Then the inequality holds:*

$$\chi(M) \geq \frac{10}{8} |\sigma(M)|$$

where  $\chi$  is the Euler characteristic.

*Strategy:* (1) We would like to propose the covering version of the inequality. Let  $X = \tilde{M}$  be the universal covering space. Then the inequality holds:

$$b_{\Gamma}^2(X) \geq \frac{10}{8} |\sigma_{\Gamma}(X)| = \frac{10}{8} |\sigma(M)|.$$

where  $b_{\Gamma}^2(X)$  is the second  $L^2$  betti number, and:

$$\sigma_{\Gamma}(X) = \dim_{\Gamma} H_{+}^2(X) - \dim_{\Gamma} H_{-}^2(X)$$

is the  $\Gamma$ -signature. It is equal to the signature over  $M$  by the Atiyah's  $\Gamma$ -index theorem.

(2) The Singer conjecture states that if  $M$  is aspherical of even dimension  $2k$ , then the  $L^2$  betti numbes vanish except middle dimension:

$$b_{\Gamma}^i(X) = 0 \quad i \neq k$$

Singer conjecture is true for Kähler hyperbolic manifolds by Gromov [G]. Other classes of aspherical manifolds are also known to be true. This is a stronger version to the Hopf conjecture which states that under the same conditions, non negativity holds:

$$(-1)^k \chi(M) \geq 0.$$

The Hopf conjecture is true in four dimensional hyperbolic manifolds by Chern [C]. This supports Singer conjecture in four dimension.

Suppose the Singer conjecture is true for an aspherical four manifold  $M$ . Then we have the equalities:

$$\begin{aligned} \chi(M) &= \chi_{\Gamma}(X) = b_{\Gamma}^0(X) - b_{\Gamma}^1(X) + b_{\Gamma}^2(X) - b_{\Gamma}^3(X) + b_{\Gamma}^4(X) \\ &= b_{\Gamma}^2(X). \end{aligned}$$

(3) We would have the inequality in conjecture 1.12 if we combine (1) and (2) above.

So far various affirmative estimates have been obtained, which would support the conjecture:

**Theorem 1.14.** (1) [Ko] *If  $M$  is an aspherical surface bundle, then the inequality  $\chi(M) \geq 2|\sigma(M)|$  holds.*

(2) [B] Suppose the intersection form of  $M$  is even, whose fundamental group is amenable or realized by  $\pi_1$  of a closed hyperbolic manifold of  $\dim \geq 3$ . Then the inequality  $\chi(M) \geq \frac{10}{8}|\sigma(M)|$  holds.

In (1), one can replace 2 by 3, which presents more stronger inequality, if moreover  $M$  admits a complex structure. The proof is rather different from our approach.

In (2), Bohr developed a very interesting argument which replies on some group theoretic properties.  $M$  is not necessarily assumed aspherical.

**Lemma 1.15.** *Suppose  $M$  is a complex surface of general type with  $c_1^2 \geq 0$ . Then the inequality  $\chi(M) \geq \frac{12}{8}|\sigma(M)|$  holds.*

The condition of  $c_1^2 \geq 0$  holds if it is minimal, or  $c_1^2 = 3c_2$  holds, where the latter case is given by the unit ball in  $\mathbb{C}^2$  divided by a discrete group action by Yau.

*Proof.* Recall the formulas  $\sigma(M) = \frac{1}{3}(c_1^2 - 2c_2)$  with  $\chi(M) = c_2(M)$ . Moreover positivity  $c_2 > 0$  holds.

Let us divide into two cases, and suppose  $\sigma > 0$  holds. Then the strict inequality:

$$\frac{10}{8}|\sigma(M)| = \frac{10}{8} \frac{1}{3}(c_1^2 - 2c_2) \leq \frac{10}{24}c_2 = \frac{10}{24}\chi(M)$$

holds by Miyaoka-Yau inequality  $c_1^2 \leq 3c_2$ .

Suppose  $\sigma(M) \leq 0$  holds. Then

$$\frac{10}{8}|\sigma(M)| = \frac{10}{8} \frac{1}{3}(2c_2 - c_1^2) \leq \frac{10}{12}c_2 = \frac{10}{12}\chi(M)$$

holds by non negativity of the Chern number. □

**Remark 1.16.** *It has been known that an oriented and definite four manifold must have diagonal and hence odd type intersection form ([D1],[D2], see [BF]).*

*An aspherical four manifold with definite form cannot satisfy the inequality in conjecture 1.12, if it exists. In fact in the case the inequality is given by:*

$$2(1 - b_1) + b_2 \geq \frac{10}{8}b_2$$

*which is the same as  $8(1 - b_1) \geq b_2 \geq 0$ . In general aspherical four manifold should satisfy the lower bound  $b_1 \geq 1$ . Then we would have a*

*contradiction to the above inequality, if  $b_1 > 1$  holds. If  $b_1 = 1$  holds, then  $b_2 = 0$  must hold.*

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## 2. MONOPOLE MAP

In section 2, we quickly review Seiberg-Witten and Bauer-Furuta theories over compact four manifolds. Then we extend their constructions over universal covering spaces of compact four manifolds. A key aspect is globally analytic structure of their fundamental groups related to their cohomology groups.

**2.1. Clifford algebras.** Let  $V$  be a real four dimensional Euclidean space, and consider its Clifford algebras  $Cl(V) = Cl_0(V) \oplus Cl_1(V)$ .

Let  $S$  be the unique complex 4 dimensional irreducible representation of  $Cl(V)$ . The complex involution is defined by:

$$\omega_{\mathbb{C}} = -e_1 e_2 e_3 e_4$$

where  $\{e_i\}_i$  is any orthonormal basis. It decomposes  $S$  into their eigen bundles as  $S = S^+ \oplus S^-$ , and induces the eigenspace decomposition:

$$Cl_0(V) \otimes \mathbb{C} \cong (Cl_0(V) \otimes \mathbb{C})^+ \oplus (Cl_0(V) \otimes \mathbb{C})^-$$

by left multiplication. It turns out that the isomorphisms hold:

$$(Cl_0(V) \otimes \mathbb{C})^{\pm} \cong \text{End}_{\mathbb{C}}(S^{\pm}).$$

Passing through the vector space isomorphism  $Cl_0(V) \cong \wedge^0 \oplus \wedge^2 \oplus \wedge^4$ , the former corresponds as follows:

$$(Cl_0(V) \otimes \mathbb{C})^+ \cong \mathbb{C} \left( \frac{1 + \omega_{\mathbb{C}}}{2} \right) \oplus (\wedge_+^2(V) \otimes \mathbb{C})$$

where the self-dual form corresponds to the trace free part. Then for any vector  $v \in S^+$ ,  $v \otimes v^* \in \text{End}(S^+)$  satisfy:

$$\sigma(v) \equiv v \otimes v^* - \frac{|v|^2}{2} \text{id} \in \wedge_+^2(V) \otimes \mathbb{C}.$$

**2.2. Monopole map over compact four manifolds.** Let  $M$  be an oriented compact Riemannian four manifold equipped with a  $\text{spin}^c$  structure. Let  $S^{\pm}$  and  $L$  be the Hermitian rank 2 bundles and the determinant bundle respectively.

Let  $A_0$  be a  $U(1)$  connection on  $L$ . With a Riemannian metric on  $M$ , it induces a  $\text{spin}^c$  connection and the associated Dirac operator  $D_{A_0}$  on  $S^{\pm 1}$ . Fix a large  $k \geq 2$  and consider the configuration space:

$$\mathfrak{D} = \{(A_0 + a, \psi) : a \in L_k^2(M; \Lambda^1 \otimes i\mathbb{R}), \psi \in L_k^2(S^+)\}.$$

Then we have the Seiberg-Witten map:

$$\begin{aligned} F : \mathfrak{D} &\rightarrow L_{k-1}^2(M; S^- \oplus \Lambda_+^2 \otimes i\mathbb{R}), \\ (A_0 + a, \psi) &\rightarrow (D_{A_0+a}(\psi), F_{A_0+a}^+ - \sigma(\psi)). \end{aligned}$$

Notice that the space of connections is independent of choice of  $A_0$  as far as  $M$  is compact.

There is symmetry  $\mathfrak{G}_* \equiv L_{k+1}^2(M; S^1)_*$  which acts on  $\mathfrak{D}$  by the group of based automorphisms with the identity at  $*$  on the  $\text{spin}^c$  bundle. The action of the gauge group  $g$  on the spinors are the standard one, and on 1 form is given by:

$$a \rightarrow a + g^{-1}dg$$

while trivial one both 0 and self-dual 2 forms.

It follows that  $F$  is equivariant with respect to the gauge group action, and hence the gauge group acts on the zero set:

$$\tilde{\mathfrak{M}} = \{(A_0 + a, \psi) \in \mathfrak{D} : F(A_0 + a, \psi) = 0\}.$$

Moreover the quotient space  $\mathfrak{B} \equiv \mathfrak{D}/\mathfrak{G}_*$  is Hausdorff.

**Definition 2.1.** *The based Seiberg-Witten moduli space is given by the quotient space:*

$$\mathfrak{M} = \tilde{\mathfrak{M}}/\mathfrak{G}_*.$$

Any connection  $A_0 + a$  with  $a \in L_k^2(M; \Lambda^1 \otimes i\mathbb{R})$  can be assumed to satisfy  $\text{Ker } d^*(a) = 0$  after gauge transform. Such gauge group element is unique, since it is based so that locally constant functions cannot appear. The slice map is given by the restriction:

$$F : L_k^2(M; S^+) \oplus (A_0 + \text{Ker } d^*) \rightarrow L_{k-1}^2(M; S^- \oplus \Lambda_+^2 \otimes i\mathbb{R})$$

whose zero set consists of the based moduli space equipped with natural  $S^1$  action:

$$\tilde{\mathfrak{M}}/\mathfrak{G}_* = F^{-1}(0) \subset L_k^2(M; S^+) \oplus (A_0 + \text{Ker } d^*).$$

Let us list some of the remarkable properties of the Seiberg-Witten moduli space for convenience.

(1) The infinitesimal model of the moduli space is given by the elliptic complex:

$$\begin{aligned} 0 \rightarrow L_{k+1}^2(M; i\mathbb{R}) &\rightarrow L_k^2(M; \Lambda^1 \otimes i\mathbb{R} \oplus S^+) \\ &\rightarrow L_{k-1}^2(M; \Lambda_+^2 \otimes i\mathbb{R} \oplus S^-) \rightarrow 0 \end{aligned}$$

where the first map is given by  $a \rightarrow (2da, -a\psi)$  and the second is by:

$$\begin{pmatrix} d^+ & -D\sigma_\psi \\ \frac{1}{2}\psi & D_A \end{pmatrix}$$

at  $(A, \psi)$ . So the formal dimension is given by:

$$\text{ind } F = \text{ind } AHS + \text{ind } D_{A_0}$$

where the right hand side consists of sum of the indices of the of AHS complex and of the Dirac operator.

(2) The moduli space consists of only the trivial solution when  $M$  admits a positive scalar curvature.

(3) The moduli space is always compact or empty.

(4) A part of the linearized map:

$$0 \rightarrow L_{k+1}^2(M) \rightarrow L_k^2(M; \Lambda^1) \rightarrow L_{k-1}^2(M; \Lambda_+^2) \rightarrow 0$$

is called the Atiyah-Hitchin-Singer complex. Let us compute the first cohomology group:

**Lemma 2.1.** *Let  $M$  be a closed four manifold. The first cohomology of the AHS complex:*

$$H^1(M) = \text{Ker } d^+ / \text{im } d$$

*is isomorphic to the 1st de Rham cohomology.*

*Proof.* There is a canonical linear map  $H_{dR}^1(M) \rightarrow H^1(M)$ .

Suppose  $d^+(a) = 0$  holds. We verify that  $d(a) = 0$  also holds. In fact we have the equalities by Stokes theorem:

$$\begin{aligned} 0 &= \int_M d(a) \wedge d(a) = \int_M d^+(a) \wedge d^+(a) + \int_M d^-(a) \wedge d^-(a) \\ &= \int_M d^-(a) \wedge d^-(a) = -\|d^-(a)\|_{L^2}^2 \end{aligned}$$

So the inverse linear map  $H^1(M) \rightarrow H_{dR}^1(M)$  exists.  $\square$

Let us introduce the monopole map as below:

$$\begin{aligned} \mu : \text{Conn} \times (\Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M)) &\rightarrow \\ &\text{Conn} \times (\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)) \\ (A, \phi, a, f) &\rightarrow (A, D_{A+a}\phi, F_{A+a}^+ - \sigma(\phi), d^*(a) + f, a_{\text{harm}}) \end{aligned}$$

where  $\text{Conn}$  is the space of  $\text{spin}^c$  connections.  $\mu$  is equivariant with respect to the gauge group action. The subspace  $A + \ker(d) \subset \text{Conn}$  is invariant under the free action of the based gauge group, and its quotient is isomorphic to the isomorphism classes of the flat bundles:

$$\text{Pic}(M) = H^1(M; \mathbb{R}) / H^1(M; \mathbb{Z}).$$



Now we denote the quotient spaces:

$$\mathfrak{A} \equiv (A + \ker(d)) \times (\Gamma(S^+) \oplus \Omega^1(X) \oplus H^0(M))/\mathfrak{G}_*,$$

$$\mathfrak{C} \equiv (A + \ker(d)) \times (\Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M))/\mathfrak{G}_*.$$

Then the monopole map descends to the fibered map over  $Pic(M)$ :

$$\mu = \tilde{\mu}/\mathfrak{G}_0 : \mathfrak{A} \rightarrow \mathfrak{C}.$$

By completion, this extends to a smooth map between Sobolev spaces. Given a trivialization  $\mathfrak{C}_{k-1} = Pic(M) \times H_{k-1}$ , one obtains the monopole map by composition with the projection:

$$\mu : \mathfrak{A}_k \rightarrow \mathfrak{C}_{k-1} = Pic(M) \times H_{k-1} \rightarrow H_{k-1}.$$

There is a finite dimensional reduction of a strongly proper map in our sense, which allows one to define *degree*. Bauer-Furuta theory verifies that the monopole map  $\mu : \mathfrak{A}_k \rightarrow H_{k-1}$  is strongly proper when a four manifold is compact, which allows one to give an element in  $S^1$  equivariant stably co-homotopy group. There is a natural homomorphism from the  $S^1$  equivariant stably co-homotopy group to integer. Its image coincides with the Seiberg-Witten invariant if  $b^+ > b^1 + 1$  holds [BF].

**2.3. Seiberg-Witten map on the universal covering space.** Let  $(M, g)$  be a smooth and closed Riemannian four manifold equipped with a  $\text{spin}^c$  structure. Denote its universal covering space and fundamental group by  $X = \tilde{M}$  and  $\Gamma$  respectively. We equip with the lift of the metric on  $X$ . The spinor bundle  $S = S^+ \oplus S^-$  over  $M$  is also lifted as  $\tilde{S} = \tilde{S}^+ \oplus \tilde{S}^-$  over  $X$ .

Let  $L_k^2(M)$  be the Sobolev spaces over  $M$ . Their norms can be lifted over  $X$  so that they are  $\Gamma$  invariant. Later on we assume such property.

Let  $A_0$  be a  $\text{spin}^c$  connection over  $M$ , and  $(A_0, \psi_0)$  be a smooth solution to the Seiberg-Witten equation so that the equalities hold:

$$D_{A_0}(\psi_0) = 0, \quad F_{A_0}^+ - \sigma(\psi_0) = 0.$$

Let us denote its lift by  $(\tilde{A}_0, \tilde{\psi}_0)$  over  $X$ , and put:

$$\sigma(\tilde{\psi}_0, \psi) \equiv \sigma(\tilde{\psi}_0 + \psi) - \sigma(\tilde{\psi}_0)$$

$$D_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a) = D_{\tilde{A}_0 + a}(\tilde{\psi}_0 + \psi) - D_{\tilde{A}_0}(\tilde{\psi}_0) = a(\tilde{\psi}_0 + \psi) + D_{\tilde{A}_0}(\psi).$$

**Lemma 2.2.** *For  $k \geq 1$ , the above maps define the continuous maps:*

$$\begin{aligned} \sigma(\tilde{\psi}_0, \cdot) : L_k^2((X, g); \tilde{S}^+) &\rightarrow L_{k-1}^2((X, g); \Lambda_+^2 \otimes i\mathbb{R}) \\ D_{\tilde{A}_0, \tilde{\psi}_0} : L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow L_{k-1}^2((X, g); \tilde{S}^-) \end{aligned}$$

If moreover  $k \geq 3$ , then it defines the continuous map:

$$\sigma(\tilde{\psi}_0, \cdot) : L_k^2((X, g); \tilde{S}^+) \rightarrow L_k^2((X, g); \Lambda_+^2 \otimes i\mathbb{R}).$$

*Proof.* Notice the equality:

$$\sigma(\tilde{\psi}_0, \psi) = \tilde{\psi}_0 \otimes \psi^* + \psi \otimes \tilde{\psi}_0^* - \langle \tilde{\psi}_0, \psi \rangle \text{id} + \sigma(\psi).$$

Since  $\psi_0 \in C^\infty(M; S^+)$  is smooth, there is a constant  $C$  such that the estimates hold:

$$\|\sigma(\tilde{\psi}_0, \psi)\|_{L_{k-1}^2} \leq C\|\psi\|_{L_{k-1}^2} + \|\sigma(\psi)\|_{L_{k-1}^2}.$$

Let us consider the last term. Notice the estimate with  $l \leq k-1$ :

$$\begin{aligned} \|\nabla^l(\psi \otimes \psi^*)\|_{L^2} &= \|\Sigma_{\alpha+\beta=l} \nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\|_{L^2} \\ &\leq \Sigma_{\alpha+\beta=l} \|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\|_{L^2} \end{aligned}$$

It follows from corollary 3.3(2) below that we obtain the estimates:

$$\begin{aligned} \|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\|_{L^2(X)}^2 &= \Sigma_{\gamma \in \Gamma} \|\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)\|_{L^2(\gamma(K))}^2 \\ &\leq \Sigma_{\gamma \in \Gamma} \|\nabla^\alpha(\psi)\|_{L^4(\gamma(K))}^2 \|\nabla^\beta(\psi^*)\|_{L^4(\gamma(K))}^2 \\ &\leq C \Sigma_{\gamma \in \Gamma} \|\nabla^\alpha(\psi)\|_{L_1^2(\gamma(K))}^2 \|\nabla^\beta(\psi^*)\|_{L_1^2(\gamma(K))}^2 \\ &\leq C \|\psi\|_{L_k^2(X)}^4 \end{aligned}$$

where  $K \subset X$  is a fundamental domain of the covering.

In particular we obtain the estimate  $\|\sigma(\psi)\|_{L_{k-1}^2} \leq C\|\psi\|_{L_k^2}^2$ , and hence in total we obtain the estimate:

$$\|\sigma(\tilde{\psi}_0, \psi)\|_{L_{k-1}^2} \leq C(\|\psi\|_{L_{k-1}^2} + \|\psi\|_{L_k^2}^2).$$

The estimate for  $D_{\tilde{A}_0, \tilde{\psi}_0}$  is obtained by the same way.

Now suppose  $k \geq 3$ , and consider  $\nabla^\alpha(\psi) \otimes \nabla^\beta(\psi^*)$  with  $\alpha + \beta = l \leq k$ . Suppose both  $\alpha$  and  $\beta$  are less than or equal to  $k-1$ . Then by the same argument as above its  $L^2$  norm is bounded by  $C\|\psi\|_{L_k^2(X)}^4$ . Next suppose  $\alpha = k \geq 3$  and hence  $\beta = 0$ . Then there is a constant  $C$  with  $\|\psi\|_{C^0(\gamma(K))} \leq C\|\psi\|_{L_k^2(\gamma(K))}$  by lemma 3.2(2) below. So we obtain the estimates:

$$\begin{aligned} \|\nabla^k(\psi) \otimes \psi^*\|_{L^2(X)}^2 &= \Sigma_{\gamma \in \Gamma} \|\nabla^k(\psi) \otimes \psi^*\|_{L^2(\gamma(K))}^2 \\ &\leq \Sigma_{\gamma \in \Gamma} \|\psi\|_{L_k^2(\gamma(K))}^2 \|\psi^*\|_{C^0(\gamma(K))}^2 \\ &\leq C\|\psi\|_{L_k^2(\gamma(K))}^2 \|\psi^*\|_{L_k^2(\gamma(K))}^2 \leq C\|\psi\|_{L_k^2(X)}^4. \end{aligned}$$

□

Later on we will choose a large  $k \gg 1$  otherwise stated.

**Definition 2.2.** *The covering Seiberg-Witten map at the base  $(A_0, \psi_0)$  is given below:*

$$\begin{aligned} F_{\tilde{A}_0, \tilde{\psi}_0} : L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \\ (\psi, a) &\rightarrow (D_{\tilde{A}_0+a}(\tilde{\psi}_0 + \psi), F_{\tilde{A}_0+a}^+ - \sigma(\tilde{\psi}_0 + \psi) - \sigma(\tilde{\psi}_0)) \\ &= (D_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), d^+(a) - \sigma(\tilde{\psi}_0, \psi)) \end{aligned}$$

The gauge group  $\mathfrak{G}_{k+1}$  over the  $\text{spin}^c$  bundle is given by:

$$\mathfrak{G}_{k+1} = \exp(L_{k+1}^2(X; i\mathbb{R}))$$

which is based at infinity, and admits structure of a Hilbert commutative Lie group for  $k \geq 3$  (see corollary 3.3(1) below). The action is given by:

$$(A, \psi) \rightarrow ((\det g)^*(A), S^+(g^{-1})(\psi))$$

for  $g \in \mathfrak{G}_{k+1}$ . Notice that  $\det \sigma = \sigma^2$ , where we regard  $\sigma : X \rightarrow S^1 \subset \mathbb{C}$ .

**Lemma 2.3.** *The gauge group acts on the covering Seiberg-Witten map at the base  $(A_0, \psi_0)$  equivariantly.*

*Proof.* Let  $g = \exp(if)$  with  $f \in L_{k+1}^2(X; i\mathbb{R})$ . Then the conclusion follows from the equality:

$$(\det g)^*(\tilde{A}_0) - \tilde{A}_0 = 2idf \in L_k^2(X; i\mathbb{R}).$$

□

**2.3.1. Reducible case.** The covering Seiberg-Witten map becomes simpler if one uses a reducible solution  $(A_0, 0)$  as the base, in which case  $A_0$  satisfies the ASD equation  $F_{A_0}^+ = 0$ . Then it is given by:

$$\begin{aligned} F : L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \\ (\psi, a) &\rightarrow (D_{\tilde{A}_0+a}(\psi), d^+(a) - \sigma(\psi)) \end{aligned}$$

Notice that the moduli space of  $U(1)$  ASD connections is given by the space of harmonic anti self-dual 2 forms on  $M$  whose cohomology class is given and integral.

**2.3.2. Equivariant gauge fixing.** Let us say that AHS complex is *closed*, if the differentials:

$$0 \rightarrow L_{k+1}^2(X) \rightarrow L_k^2(X; \Lambda^1) \rightarrow L_{k-1}^2(X; \Lambda_+^2) \rightarrow 0$$

have closed range.

**Lemma 2.4.** *Suppose the AHS complex is closed.. Then the first cohomology group  $H^1(X) = \text{Ker } d^+ / \text{im } d$  is isomorphic to the  $L^2$  1st deRham cohomology.*

*Proof.* This follows by the same argument as lemma 2.1. □

**Remark 2.5.** *Later we see several classes of universal covering spaces whose AHS complexes is closed. Actually in many cases such property depends only on the large scale analytic property of their fundamental groups.*

Let us state the equivariant gauge fixing:

**Proposition 2.6.** *Suppose the AHS complex is closed.*

*Then there is a global and equivariant gauge fixing so that one can restrict to the slice for the covering Seiberg-Witten map:*

$$\begin{aligned} F_{\tilde{A}_0, \tilde{\psi}_0} : L_k^2(X; \tilde{S}^+) \oplus L_k^2(X; \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* \\ \rightarrow L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}). \end{aligned}$$

*Proof. Step 1:* Let  $A = \tilde{A}_0 + a$  with  $a \in L_k^2(X; i\mathbb{R})$ . We verify that there is  $\sigma \in \mathfrak{G}_{k+1}$  such that  $(\det \sigma)^*(A) \equiv \tilde{A}_0 + a'$  satisfies the equality  $d^*(a') = 0$  with the estimate:

$$\|a'\|_{L_k^2(X)} \leq C(\|d^+(a')\|_{L_{k-1}^2(X)} + \|a'_{\text{harm}}\|)$$

for some constant  $C$  independent of  $A_0$ .

Notice that when one finds such a constant for some  $A_0$ , then it holds for any choice of the base, since the Seiberg-Witten moduli space is compact over the base compact manifold  $M$  (see 2.2).

It follows from the assumption that there is a bounded linear map:

$$\Delta^{-1} : d^*(L_k^2(X; \Lambda^1 \otimes i\mathbb{R})) \rightarrow L_{k+1}^2(X; i\mathbb{R})$$

which inverts the Laplacian. Let us put:

$$s_0 = -\frac{1}{2}\Delta^{-1}(d^*(a)) \in L_{k+1}^2(X; i\mathbb{R})$$

and  $\sigma_0 = \exp(s_0) \in \mathfrak{G}_{k+1}$ . For  $a' = a + 2\sigma_0^{-1}d\sigma_0$ , we have:

$$(\det(\sigma_0))^*(A) = \tilde{A}_0 + a'$$

with the equality  $d^*(a') = 0$ .

**Step 2:** Let us consider:

$$d^* \oplus d^+ : L_k^2(X; \Lambda^1 \otimes i\mathbb{R}) \rightarrow L_{k-1}^2(X; (\Lambda^0 \oplus \Lambda_+^2) \otimes i\mathbb{R})$$

Its kernel is the space of harmonic one forms, and decompose  $a' = h + b$ , where  $h$  is the harmonic form and  $b$  lies in the orthonormal subspace. Then it follows from closedness that there is a bound:

$$\|b\|_{L_k^2} \leq C(\|d^*(b)\|_{L_{k-1}^2} + \|d^+(b)\|_{L_{k-1}^2}) = C\|d^+(b)\|_{L_{k-1}^2}.$$

Moreover  $d^+(b) = F_A^+ - F_{\tilde{A}_0}^+ = d^+(a')$  holds. So we have the estimates:

$$\|a'\|_{L_k^2}^2 \leq C(\|d^+(a')^+\|_{L_{k-1}^2} + \|a'_{harm}\|).$$

□

**2.3.3. Covering Seiberg-Witten moduli space.** Let us consider the closed subset:

$$\begin{aligned} \mathfrak{M}(A_0, \psi_0) &\equiv F_{\tilde{A}_0, \tilde{\psi}_0}^{-1}(0) \\ &\subset L_k^2(X; \tilde{S}^+) \oplus L_k^2(X; \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^*. \end{aligned}$$

It is non empty since  $[(A_0, \psi_0)]$  is an element in it. If  $F_{\tilde{A}_0, \tilde{\psi}_0}$  is regular so that its differential is surjective on  $\mathfrak{M}(A_0, \psi_0)$ , then it is a regular manifold equipped with the induced  $\Gamma$  action. Its  $\Gamma$ -dimension is equal to:

$$\text{ind } D_{A_0} + \chi_{AHS}$$

where the former is the index of  $D_{A_0}$  and the second is the AHS-Euler characteristic on  $M$ .

Notice that if an element  $g \in \Gamma$  is infinite cyclic, then  $g$ -action is free except the origin  $[(A_0, \psi_0)]$ .

Choose any  $x, y \in F_{\tilde{A}_0, \tilde{\psi}_0}^{-1}(0)$ , and consider its differential:

$$\begin{aligned} d(F_{\tilde{A}_0, \tilde{\psi}_0})_x : L_k^2(X; \tilde{S}^+) \oplus L_k^2(X; \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* \\ \rightarrow L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}). \end{aligned}$$

Let us denote  $d(F_{\tilde{A}_0, \tilde{\psi}_0})_x$  just by  $dF_x$  in short.

**Lemma 2.7.** (1) For  $x = (a, \psi)$ , the following formula holds:

$$dF_x(c, \xi) = (D_{\tilde{A}_0+a}(\xi) + c(\tilde{\psi}_0 + \psi), d^+(c) - (\tilde{\psi}_0 + \psi) \otimes \xi^* - \xi \otimes (\tilde{\psi}_0 + \psi)^*).$$

(2) Let  $k \geq 3$ . The difference  $dF_x - dF_y$  is compact.

*Proof.* We have the formula:

$$\begin{aligned} dF_x(c, \xi) &= \frac{d}{dt}(D_{\tilde{A}_0+a+tc}(\tilde{\psi}_0 + \psi + t\xi), d^+(c) - \sigma(\tilde{\psi}_0, \psi + t\xi))_{t=0} \\ &= (D_{\tilde{A}_0+a}(\xi) + c(\tilde{\psi}_0 + \psi), d^+(c) - (\tilde{\psi}_0 + \psi) \otimes \xi^* - \xi \otimes (\tilde{\psi}_0 + \psi)^*). \end{aligned}$$

Let and  $y = (b, \phi)$ . Their difference is given by:

$$(dF_x - dF_y)(c, \xi) = (a - b)\xi + c(\psi - \phi), -(\psi - \phi) \otimes \xi^* - \xi \otimes (\psi - \phi)^*$$

Since all  $a, b, \phi, \psi, \xi \in L_k^2$ , Their products all lies in  $L_k^2$  by corollary 3.3. If  $a, b, \psi, \phi$  all have compact support, then compactness follows from the Sobolev multiplication with Rellich lemma. In general they can be approximated by compactly supported smooth functions, and so the differences are still compact.  $\square$

**2.4. Covering monopole map.** Let  $M$  be a compact oriented four manifold equipped with a  $\text{spin}^c$  structure, and  $X = \tilde{M}$  be its universal covering space with  $\pi_1(M) = \Gamma$ . Let  $H^1(X)$  ( $\bar{H}^1(X)$ ) be the (reduced)  $L^2$  cohomology. The reduced  $L^2$  cohomology coincides with the unreduced one  $\bar{H}^*(X) = H^*(X)$  when AHS complex is closed.

Let  $(A_0, \psi)$  be a solution to the Seiberg-Witten equation over  $M$ , and denote its lift by  $(\tilde{A}_0, \tilde{\psi})$  over  $X$ .

**Definition 2.3.** *The covering monopole map at the base  $(A_0, \psi_0)$  is the  $\mathfrak{G}_{k+1} \rtimes \Gamma$  equivariant map given below:*

$$\begin{aligned} \tilde{\mu} : L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) &\rightarrow \\ L_{k-1}^2((X, g); \tilde{S}^- \oplus (\Lambda_+^2 \oplus \Lambda^0) \otimes i\mathbb{R}) &\oplus \bar{H}^1(X) \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), d^*(a), [a]) \end{aligned}$$

where  $[\ ]$  is the orthogonal projection to the reduced cohomology group.

**Remark 2.8.** *Even if the first deRham cohomology group  $H^1(X; \mathbb{R}) = 0$  vanishes, still the first reduced cohomology may survive. For an element in the latter cohomology, there associates a ‘gauge group’ whose action on the element can eliminate it. Such gauge group of course does not lie in the  $L^2$  Sobolev space, whose behavior at infinity seems rather complicated so that they will ‘move’ quite ‘slowly’ at infinity.*

**Lemma 2.9.** *Suppose the AHS complex is closed. Then the covering monopole map restricts over the slice which is  $\Gamma$  equivariant map:*

$$\begin{aligned} \tilde{\mu} : L_k^2((X, g); \tilde{S}^+) \oplus L_k^2((X, g); \Lambda^1 \otimes i\mathbb{R}) &\cap \text{Ker } d^* \rightarrow \\ L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) &\oplus H^1(X) \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), [a]) \end{aligned}$$

The linearized map is  $\Gamma$ -Fredholm whose  $\Gamma$  index coincides with:

$$\begin{aligned} \dim_{\Gamma} d\tilde{\mu} &= \text{ind } D - (b_0(M) - b_1(M) + b_2^+(M)) - \dim_{\Gamma} H^1(X) \\ &= \text{ind } D - \dim_{\Gamma} H_+^2(X). \end{aligned}$$

where  $\text{ind } D$  is the index of the Dirac operator over  $M$ .

*Proof.* The former follows from proposition 2.6. The latter follows from the Atiyah's  $\Gamma$ -index theorem.  $\square$

**Remark 2.10.** *The  $\Gamma$ -dimension is topological invariant of the base manifold  $M$  when one of  $H^1(X) = 0$  or  $H_+^2(X) = 0$  hold.*

*If  $M$  is compact and aspherical, Singer conjecture states that  $L^2$  cohomology should vanish except middle dimension, where in our case of four manifolds, only the second  $L^2$  cohomology is able to survive and  $H^1(X)$  should vanish. It has been verified for many classes of compact aspherical manifolds whose fundamental groups possess 'hyperbolic' structure [Gr1].*

### 3. $L^p$ ANALYSIS AND ESTIMATES ON SOBOLEV SPACES

**3.1. Sobolev spaces over covering spaces.** Let  $l : C^\infty(M) \rightarrow C^\infty(M)$  be a first order elliptic differential operator over a compact Riemannian manifold, and lift it over the universal covering space  $X = \tilde{M}$ . Let  $\langle u, v \rangle = \int_X (u(x), v(x)) \text{vol}$  be the lift of the  $L^2$  inner product over  $X$ , which is  $\Gamma$  invariant, and  $l^*$  be its formal adjoint operator. We will use the Sobolev norms over  $X$  by:

$$\begin{aligned} \langle u, v \rangle_{L_1^2} &= \langle u, v \rangle + \langle l(u), l(v) \rangle, \\ \langle u, v \rangle_{L_2^2} &= \langle u, v \rangle + \langle l(u), l(v) \rangle + \langle l^*l(u), l^*l(v) \rangle, \\ &\dots \end{aligned}$$

whose spaces are given by taking closure of  $C_c^\infty(X)$ .

**Lemma 3.1.**  $\langle l(v), w \rangle_{L_k^2} = \langle v, l^*(w) \rangle_{L_k^2}$  hold for all  $k \geq 0$ .

*Proof.* It holds for  $k = 0$ . Suppose it holds up to  $k - 1$ . Since the equalities hold by induction:

$$\begin{aligned} \langle l(u), w \rangle_{L_k^2} &= \langle l(u), w \rangle + \langle l^*l(u), l^*(w) \rangle_{L_{k-1}^2} \\ &= \langle u, l^*(w) \rangle + \langle l(u), ll^*(w) \rangle_{L_{k-1}^2} \\ &= \langle u, l^*(w) \rangle_{L_k^2} \end{aligned}$$

the conclusion also holds for  $k$ .  $\square$

For convenience we recall the local Sobolev estimates over four dimensional spaces. By local compactness we mean that it is compact

on the restrictions of the function spaces with support on  $K$  for any compact subset  $K \subset X$ .

We refer [GT] for more detailed analysis of the Sobolev spaces.

**Lemma 3.2.** (1) *The continuous embeddings  $L_k^p \subset L_l^q$  hold locally, if both  $k \geq l$  and  $k - \frac{4}{p} \geq l - \frac{4}{q}$  hold.*

*They are locally compact, if the stronger inequalities  $k > l$  and  $k - \frac{4}{p} > l - \frac{4}{q}$  hold.*

(2) *The continuous embeddings  $L_k^p \subset C^l$  hold locally, if  $k - \frac{4}{p} > l$  hold.*

In particular it is convenient for us to check the embeddings  $L_k^2 \subset L_{k-1}^4$ .

**Corollary 3.3.** *The following local multiplications are continuous locally:*

- (1)  $L_k^2 \times L_k^2 \rightarrow L_k^2$  for  $k \geq 3$ .
- (2)  $L_k^2 \times L_k^2 \rightarrow L_{k-1}^2$  for  $k \geq 1$ .

*Proof.* Let us take  $u, v \in L_k^2$ . For  $k' \leq k$ ,

$$\nabla^{k'}(uv) = \sum_{a+b=k'} \nabla^a(u) \nabla^b(v)$$

hold. If  $0 \leq k' < k$ , then the estimates:

$$\begin{aligned} \|\nabla^a(u) \nabla^b(v)\|_{L_{loc}^2} &\leq \|\nabla^a(u)\|_{L_{loc}^4} \|\nabla^b(v)\|_{L_{loc}^4} \\ &\leq C \|\nabla^a(u)\|_{(L_1^2)_{loc}} \|\nabla^b(v)\|_{(L_1^2)_{loc}} \end{aligned}$$

hold by lemma 3.2(1). So we obtain the estimate:

$$\|uv\|_{(L_{k'}^2)_{loc}} \leq C \|u\|_{(L_k^2)_{loc}} \|v\|_{(L_k^2)_{loc}}.$$

This verifies (2).

Let us verify (1). Suppose  $3 \leq k' = k$ . Then we obtain the estimate:

$$\|\nabla^k(u)v\|_{(L^2)_{loc}} \leq C \|v\|_{C^0} \|u\|_{(L_k^2)_{loc}}$$

by lemma 3.2(2). Combining with the above case, we have verified (1).  $\square$

**3.2.  $L^p$  cohomology.** Let  $(X, g)$  be a complete Riemannian manifold. For  $p > 1$ , let  $L_k^p(X; \Lambda^m)$  be the Banach space of  $L_k^p$  differential  $m$  forms over  $X$ , and  $d$  be the differentials whose domains are  $C_c^\infty(X; \Lambda^m)$ .

Let us recall that:

(1) The (unreduced)  $L^p$  cohomology  $H^{m,p}(X)_k$  is given by:

$$\text{Ker } d : L_k^p(X, \Lambda^m) \rightarrow L_{k-1}^p(X, \Lambda^{m+1}) / \text{im } d : L_{k+1}^p(X, \Lambda^{m-1}) \rightarrow L_k^p(X, \Lambda^m)$$



(2) The reduced  $L^p$  cohomology  $\bar{H}^{m,p}(X)_k$  is given by:

$$\text{Ker } d : L_k^p(X, \Lambda^m) \rightarrow L_{k-1}^p(X, \Lambda^{m+1}) / \overline{\text{im } d} : L_{k+1}^p(X, \Lambda^{m-1}) \rightarrow L_k^p(X, \Lambda^m)$$

where  $\overline{\text{im}}$  is the closure of the image.

There is a canonical surjection  $H^{m,p}(X)_k \rightarrow \bar{H}^{m,p}(X)_k$ , and its kernel is called the torsion of  $L^p$  cohomology:

$$T_k^{m,p} \equiv \text{Ker } \{ H^{m,p}(X)_k \rightarrow \bar{H}^{m,p}(X)_k \}.$$

$d$  has closed range if and only if the torsion vanishes  $T_k^{m,p} = 0$ .

**Definition 3.1.**  $L^p$  harmonic space  $\mathfrak{H}^{m,p}(X)$  is given by:

$$\text{Ker } (d \oplus d^*) : L_k^p(X, \Lambda^m) \rightarrow L_{k-1}^p(X, \Lambda^{m+1} \oplus \Lambda^{m-1}).$$

Notice that  $\mathfrak{H}^{m,p}(X)$  is independent of choice of  $k$ . It is well known that the space  $\bar{H}^{m,2}(X)_k$  is isomorphic to  $L^2$  harmonic  $m$  forms. In particular they are independent of  $k$ .

For our case of the AHS complex, the second cohomology involves  $d^+$  rather than  $d$ .

**Lemma 3.4.** Suppose  $d : L_k^2(X; \Lambda^i) \rightarrow L_{k-1}^2(X; \Lambda^{i+1})$  have closed range for  $i = 0, 1$ . Then the composition with the self-dual projection:

$$d^+ : L_k^2(X; \Lambda^1) \rightarrow L_{k-1}^2(X; \Lambda_+^2)$$

also has closed range.

*Proof. Step 1:* Let  $W \subset H$  be a closed linear subspace. If a sequence  $w_i \in W$  weakly converge to some  $w \in H$ , then  $w \in W$ . In fact  $\langle w, h \rangle = \lim_i \langle w_i, h \rangle = 0$  for any  $h \in W^\perp$ .

Let  $H_1$  and  $H_2$  be both Hilbert spaces, and  $W \subset H_1 \oplus H_2$  be a closed linear subspace. Let us consider the projection  $P : H_1 \oplus H_2 \rightarrow H_1$ , and choose a sequence  $w_i = v_i^1 + v_i^2 \in W \subset H_1 \oplus H_2$ . Suppose the sequence  $P(w_i) = v_i^1 \in H_1$  converge to some  $v_1 \in H_1$ , and weak limit of  $w_i$  does not lie on  $W$ . Then  $\|v_i^2\| \rightarrow \infty$  must hold. In fact if  $\|v_i^2\|$  could be bounded, then  $v_i^2$  weakly converge to some  $v_2 \in H_2$ . In particular  $w_i$  weakly converge to  $v_1 + v_2$  which should lie in  $W$  as we have verified.

**Step 2:** Let us verify the conclusion for  $k = 1$ . It follows from Stokes theorem that for  $\alpha \in L_1^2(X; \Lambda^1)$ ,

$$0 = \int_X d(\alpha) \wedge d(\alpha) = \|d^+(\alpha)\|_{L^2}^2 - \|d^-(\alpha)\|_{L^2}^2$$

So we have the equality  $\|d^+(\alpha)\|_{L^2}^2 = \|d^-(\alpha)\|_{L^2}^2$ .

It follows from step 1 that if a sequence  $\alpha_i \in L_1^2(X; \Lambda^1)$  satisfy convergence  $d^+(\alpha) \rightarrow a \in L^2(X; \Lambda_+^2)$ , then  $a = d^+(\alpha)$  for some  $\alpha \in L_1^2(X; \Lambda^1)$ , otherwise  $d^-(\alpha_i)$  should diverge in  $L^2$  norm.

**Sep 3:** Let us verify  $k = 2$  case, and assume  $\alpha_i \in L_2^2(X; \Lambda^1)$  satisfy convergence  $d^+(\alpha) \rightarrow a \in L_1^2(X; \Lambda_+^2)$ . Then there is some  $\alpha \in L_1^2(X; \Lambda^1)$  with  $d^+(\alpha) = a$  by step 2.

We may assume  $d^*(\alpha) = 0$  since  $d : L_{k+1}^2(X) \rightarrow L_k^2(X; \Lambda^1)$  has closed range. Then the elliptic estimate tells  $\alpha \in L_2^2(X; \Lambda^1)$  and hence  $k = 2$  case follows.

We can proceed by induction so that the conclusion holds. □

**3.3. Examples of zero torsion  $L^2$  cohomology.** There are several cases of zero torsion  $L^p$  cohomology. See [P] for  $p \neq 2$  case.

Let us consider the case  $p = 2$ . By use of Hilbert space structure, there has been discovered many instances of zero torsion. Below let us list some of them.

**3.3.1. Kähler hyperbolic manifolds.** Let  $(M, \omega)$  be a compact Kähler manifold, and assume that the lift of the Kähler form  $\tilde{\omega}$  over the universal covering space  $X$  represents zero in the second de Rham cohomology  $H^2(X; \mathbb{R})$  so that it can be given as  $\tilde{\omega} = d(\eta)$  for some  $\eta \in C^\infty(X; \Lambda^1)$ .

**Lemma 3.5 (G).** *Suppose  $\|\eta\|_{L^\infty} < \infty$  is finite. Then the  $L^2$  de Rham differentials have closed range.*

*Moreover it satisfies the Singer conjecture.*

See 1.3 and also remark 2.11 on Singer conjecture.

**3.3.2. Zero torsion with positive scalar curvature.** Let us present an example of four manifolds with positive scalar curvature whose universal covering spaces have zero torsion.

**Lemma 3.6.** *Let  $X$  and  $Y$  be complete Riemannian manifolds of dimension 2, where  $X$  is non compact and  $Y$  is compact. Suppose the de Rham differentials have closed range over  $X$ .*

*Then the AHS complex over  $X \times Y$  also has closed range.*

The following argument is quite straightforward, and can be applied to more general cases.

*Proof. Step 1:* It follows from lemma 3.4 that  $d^+$  also has closed range if  $d$  is the case on 1 forms. So it is enough to verify closedness of  $d$  over both 0 and 1 forms.

Notice that  $C_c^\infty(X \times Y) \subset L^2(X \times Y)$  is dense and  $L^2(X \times Y) = L^2(X) \otimes L^2(Y)$  holds, where the right hand side is Hilbert space tensor product.

Let  $\{f_\lambda\}_\lambda$  and  $\{u_\lambda\}_\lambda$  be the spectral decompositions of  $L^2$  zero and one forms on  $Y$  with respect to the Laplacians with their eigenvalues  $\lambda^2$ . One can decompose as  $u_\lambda = d\alpha_\lambda + d^*\omega_\lambda$ .

**Step 2:** Let us consider the case of 0 forms. Let us decompose  $\alpha_1 = \sum_\lambda a_\lambda \otimes f_\lambda$  and  $\alpha_2 = \sum_\lambda g_\lambda \otimes b_\lambda$ . Then

$$\|\alpha\|_{L_k^2}^2 = \sum_\lambda \lambda^{2(k-j)} \sum_{j=0}^k (\|a_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2 + \|g_\lambda\|_{L_j^2}^2 \|b_\lambda\|_{L^2}^2).$$

Let  $\alpha \in L_k^2(X \times Y; \Lambda^1)$  be in the image:

$$d(L_{k+1}^2(X \times Y)) \subset (d(L_1^2(X)) \otimes L^2(Y)) \oplus (L^2(X) \otimes d(L_1^2(Y)))$$

and decompose as  $\alpha = \alpha_1 + \alpha_2 \in L_k^2(X \times Y; \Lambda^1(X) \oplus \Lambda^1(Y))$ . Notice that the right hand side is defined since both  $d(L_1^2(X))$  and  $d(L_1^2(Y))$  are closed in  $L^2$ .

Suppose both  $a_\lambda = dg_\lambda$  and  $b_\lambda = df_\lambda$  hold. Notice two properties:

$$\|df_\lambda\|_{L^2}^2 = \lambda^2 \|f_\lambda\|_{L^2}^2, \quad \|dg_\lambda\|_{L_j^2} \geq C \|g_\lambda\|_{L_{j+1}^2}.$$

Now we have the bounds:

$$\begin{aligned} \|\alpha\|_{L_k^2}^2 &= \|d \sum_\lambda g_\lambda \otimes f_\lambda\|_{L_k^2}^2 \\ &= \sum_\lambda \lambda^{2(k-j)} \sum_{j=0}^k (\|dg_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2 + \|g_\lambda\|_{L_j^2}^2 \|df_\lambda\|_{L^2}^2) \\ &= \sum_\lambda \lambda^{2(k-j)} \sum_{j=0}^k (\|dg_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2 + \lambda^2 \|g_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2) \\ &\geq C \sum_\lambda \lambda^{2(k-j)} \sum_{j=0}^k \|g_\lambda\|_{L_{j+1}^2}^2 \|f_\lambda\|_{L^2}^2 \\ &\quad + \sum_{\lambda \neq 0} \lambda^{2(k+1-j)} \sum_{j=0}^k \|g_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2 \\ &= C \sum_{j=0}^k \|g_\lambda\|_{L_{j+1}^2}^2 \|f_\lambda\|_{L_{k-j}^2}^2 + \sum_{\lambda \neq 0} \lambda^{2(k+1-j)} \sum_{j=0}^k \|g_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L^2}^2 \\ &\geq C' \sum_{j=0}^{k+1} \|g_\lambda\|_{L_j^2}^2 \|f_\lambda\|_{L_{k+1-j}^2}^2 = C' \left\| \sum_\lambda g_\lambda \otimes f_\lambda \right\|_{L_{k+1}^2}^2. \end{aligned}$$

This verifies that  $d$  has closed range on zero forms over  $X \times Y$ .

**Step 3:** Let us verify a general fact. Let  $f : H \rightarrow W = W_1 \oplus W_2$  be a linear map between Hilbert spaces, and suppose that the compositions

with the projections  $f_i : H \rightarrow W_i$  have both closed ranges. Then we claim that  $f$  itself has closed range.

To see this one may assume that  $f$  is injective, since image of projection of a closed linear subspace is also closed.

Take an element  $w = w_1 + w_2$  in the closure of  $f(H)$ , and choose  $\alpha^i \in H$  with  $f(\alpha^i) \rightarrow w$ . Both  $w_i$  lie in the closure of  $f_i(H)$ , and hence there is some  $a_1 \in H$  such that  $f_1(a_1) = w_1$  holds. One may assume that  $\beta^i = \alpha^i - a_1 \in \text{Ker } f_1$ .

Similarly one has:

$$\alpha^i = a_1 + \beta^i = a_2 + \gamma^i$$

where  $\gamma^i \in \text{Ker } f_2$ . Notice that the norms of the families  $\{\|\beta^i\|\}_i$  and  $\{\|\gamma^i\|\}_i$  are both at the same time uniformly bounded or both unbounded.

If both are uniformly bounded, then a weak limit  $\alpha$  of  $\{\alpha^i\}_i$  satisfies  $f(\alpha) = w$ .

Suppose they are unbounded. It follows from the equality:

$$\frac{\gamma^i}{\|\gamma^i\|} = \frac{a_1 - a_2}{\|\gamma^i\|} + \frac{\beta^i}{\|\gamma^i\|}$$

with  $\|\gamma^i\| \rightarrow \infty$  that  $\text{Ker } f_1 \cap \text{Ker } f_2 \neq 0$  must hold, since the first term on the right hand side converges to zero and these kernel subspaces are closed.

This verifies the claim.

**Step 4:** Let us consider the case of one forms. Let  $u \in L_k^2(X \times Y; \Lambda^0(X) \oplus \Lambda^1(Y))$  and consider  $w = du$ . Decompose

$$\begin{aligned} u &= \sum_{\lambda} f_{\lambda} \otimes u_{\lambda} \\ &= \sum_{\lambda} f_{\lambda} \otimes (\alpha_{\lambda} + d^* \omega_{\lambda}) \end{aligned}$$

where  $\alpha_{\lambda}$  are of the form  $dh_{\lambda}$  for  $\lambda \neq 0$ . Then:

$$\begin{aligned} du &= \sum_{\lambda} df_{\lambda} \otimes u_{\lambda} + \sum_{\lambda} f_{\lambda} \otimes du_{\lambda} \\ &\in L_{k-1}^2(X \times Y; \Lambda^1(X) \otimes \Lambda^1(Y)) \oplus L_{k-1}^2(X \times Y; \Lambda^0(X) \otimes \Lambda^2(Y)) \\ &= \sum_{\lambda} df_{\lambda} \otimes \alpha_{\lambda} + \sum_{\lambda} df_{\lambda} \otimes d^* \omega_{\lambda} + \sum_{\lambda} f_{\lambda} \otimes dd^* \omega_{\lambda}. \end{aligned}$$

It is enough to see closeness of the differential on each term above from step 3. Let us consider the restriction of the differential to the

first term:

$$\begin{aligned} d : u &= \sum_{\lambda} f_{\lambda} \otimes \alpha_{\lambda} \in L_k^2(X \times Y; \Lambda^0(X) \otimes \Lambda^1(Y)) \\ &\rightarrow du = \sum_{\lambda} df_{\lambda} \otimes \alpha_{\lambda} \in L_{k-1}^2(X \times Y; \Lambda^1(X) \otimes \Lambda^1(Y)) \end{aligned}$$

This has closed range since  $d$  is closed on  $X$ , and hence have a bound  $\|df_{\lambda}\|_{L_{k-1}^2} \geq C\|f_{\lambda}\|_{L_k^2}$ .

Let us verify that closeness of the differential on the rest terms:

$$\begin{aligned} d : u &= \sum_{\lambda} f_{\lambda} \otimes d^* \omega_{\lambda} \in L_k^2(X \times Y; \Lambda^0(X) \otimes \Lambda^1(Y)) \\ &\rightarrow du = \sum_{\lambda} df_{\lambda} \otimes d^* \omega_{\lambda} + f_{\lambda} \otimes dd^* \omega_{\lambda} \\ &\in L_{k-1}^2(X \times Y; \Lambda^1(X) \otimes \Lambda^1(Y) \oplus \Lambda^0(X) \otimes \Lambda^2(Y)) \end{aligned}$$

Notice the equalities:

$$\|dd^* \omega_{\lambda}\|_{L^2}^2 = \langle dd^* \omega_{\lambda}, dd^* \omega_{\lambda} \rangle = \langle d^* dd^* \omega_{\lambda}, d^* \omega_{\lambda} \rangle = \lambda^2 \|d^* \omega_{\lambda}\|_{L^2}^2.$$

Then we have the estimates:

$$\begin{aligned} \|du\|_{L_{k-1}^2}^2 &= \sum_{\lambda} \lambda^{2(k-1-j)} \sum_{j=0}^{k-1} \|df_{\lambda}\|_{L_j^2}^2 \|d^* \omega_{\lambda}\|_{L^2}^2 \\ &\quad + \sum_{\lambda} \lambda^{2(k-1-j)} \sum_{j=0}^{k-1} \|f_{\lambda}\|_{L_j^2}^2 \|dd^* \omega_{\lambda}\|_{L^2}^2 \\ &= \sum_{\lambda} \lambda^{2(k-1-j)} \sum_{j=0}^{k-1} \|df_{\lambda}\|_{L_j^2}^2 \|d^* \omega_{\lambda}\|_{L^2}^2 + \sum_{\lambda} \lambda^{2(k-j)} \sum_{j=0}^{k-1} \|f_{\lambda}\|_{L_j^2}^2 \|d^* \omega_{\lambda}\|_{L^2}^2 \\ &\geq C \sum_{\lambda} \lambda^{2(k-1-j)} \sum_{j=0}^{k-1} \|f_{\lambda}\|_{L_{j+1}^2}^2 \|d^* \omega_{\lambda}\|_{L^2}^2 + \sum_{\lambda} \lambda^{2(k-j)} \sum_{j=0}^{k-1} \|f_{\lambda}\|_{L_j^2}^2 \|d^* \omega_{\lambda}\|_{L^2}^2 \\ &\geq C' \|u\|_{L_k^2}^2. \end{aligned}$$

**Step 5:** Let us consider the case:

$$u = \sum_{\lambda} v_{\lambda} \otimes g_{\lambda} \in L_k^2(X \times Y; \Lambda^1(X) \otimes \Lambda^0(Y)).$$

Let us decompose  $v_{\lambda} = \alpha_{\lambda} + d^* \omega_{\lambda}$  with  $d\alpha_{\lambda} = 0$ .

By a similar argument as step 4 the compositions of the projections with the differential:

$$\begin{aligned} \text{pr} \circ d &: L_k^2(X \times Y; \Lambda^1(X) \otimes \Lambda^0(Y)) \rightarrow L_k^2(X \times Y; \Lambda^2(X) \otimes \Lambda^0(Y)), \\ \text{pr}' \circ d &: L_k^2(X \times Y; \Lambda^1(X) \otimes \Lambda^0(Y)) \rightarrow L_k^2(X \times Y; \Lambda^1(X) \otimes \Lambda^1(Y)) \end{aligned}$$

have both closed range.

**Step 6:** Let us consider the final case, which is a linear combinations of step 4 and 5. Let us consider the case:

$$\begin{aligned} u &= \sum_{\lambda} v_{\lambda} \otimes g_{\lambda} + f_{\lambda} \otimes u_{\lambda} \\ &\in L_k^2(X \times Y; \Lambda^1(X) \otimes \Lambda^0(Y) \oplus \Lambda^0(X) \otimes \Lambda^1(Y)). \end{aligned}$$

Again, one can obtain closeness of the differential by checking the property for each degree of the differential forms on  $\Lambda^*(X) \otimes \Lambda^*(Y)$  by step 3. The only remaining case to be checked is closeness of the image in  $\Lambda^1(X) \otimes \Lambda^1(Y)$ .

Let us consider the differential:

$$d : u = \sum_{\lambda} \alpha_{\lambda} \otimes g_{\lambda} + f_{\lambda} \otimes \beta_{\lambda} \rightarrow \sum_{\lambda} \alpha_{\lambda} \otimes dg_{\lambda} + df_{\lambda} \otimes \beta_{\lambda}$$

where both  $d\alpha_{\lambda} = 0$  and  $d\beta_{\lambda} = 0$  hold.

One can check closedness of  $d$  on the restrictions:

$$u = \sum_{\lambda} h_{\lambda} \otimes g_{\lambda} + f_{\lambda} \otimes h'_{\lambda}$$

where both  $h_{\lambda}$  and  $h'_{\lambda}$  are harmonic one forms.

Let us consider the rest case:

$$d : u = \sum_{\lambda} dg'_{\lambda} \otimes g_{\lambda} + f_{\lambda} \otimes df'_{\lambda} \rightarrow \sum_{\lambda} dg'_{\lambda} \otimes dg_{\lambda} + df_{\lambda} \otimes df'_{\lambda}$$

But its image coincides with the one:

$$d : u = \sum_{\lambda} f'_{\lambda} \otimes df_{\lambda} \rightarrow \sum_{\lambda} df'_{\lambda} \otimes df_{\lambda}$$

which has also closed range by a similar argument as step 2.

This finishes all the cases. □

In particular,  $\Sigma_g \times S^2$  satisfy two conditions that the AHS complexes over their universal overing spaces have closed range and the Dirac operators are invertible for all  $g \geq 2$ .

Let us compute the  $\Gamma$  indices of the AHS complex over  $\mathbf{H} \times S^2$  with  $\Gamma = \pi_1(\Sigma_g)$  actions. Let us denote by  $H_\Gamma^*(\Sigma_g \times S^2)$  as the  $L^2$  cohomology group  $H^*(\mathbf{H}^2 \times S^2)$  equipped with  $\Gamma = \pi_1(\Sigma_g)$  actions.

**Lemma 3.7.** *For any  $g \geq 2$ , the  $L^2$  cohomology  $H_\Gamma^*(\Sigma_g \times S^2)$  are zero for  $*$  = 0, 2 and satisfy the formulas:*

$$\dim_\Gamma H_\Gamma^1(\Sigma_g \times S^2) = 2g - 2.$$

*Proof. Step 1:* A harmonic function over complete non compact manifold is zero, and hence  $H_\Gamma^0(\Sigma_g) = 0$ . By Hodge duality, any harmonic 2 forms on  $\mathbf{H}^2$  are also zero. It follows from the Atiyah's  $\Gamma$  index theorem that the formula holds:

$$\dim_\Gamma H_\Gamma^1(\Sigma_g) = 2g - 2.$$

Since  $H_\Gamma^1(\Sigma_g)$  is isomorphic to the space of  $L^2$  harmonic one forms, we have the estimate:

$$\dim_\Gamma H_\Gamma^1(\Sigma_g \times S^2) \geq 2g - 2.$$

**Step 2:** We claim that the above estimate is actually equal. Let  $\alpha \in L^2(\mathbf{H}^2 \times S^2; \Lambda^1)$  be an  $L^2$  harmonic one form, and decompose:

$$\alpha = \alpha_1 + \alpha_2$$

with respect to  $\Lambda^1(\mathbf{H}^2 \times S^2) \cong \Lambda^1(\mathbf{H}^2) \oplus \Lambda^1(S^2)$ . Notice that each component  $\alpha_i \in L_k^2(\mathbf{H}^2 \times S^2; \Lambda^1)$  hold for any  $k \geq 0$ .

It follows from  $d(\alpha) = 0$  that

$$d_1(\alpha_1) = 0, \quad d_2(\alpha_2) = 0$$

hold where  $d_i$  are the differentials with respect to  $i$ -th components. Since  $H^1(S^2) = 0$  holds, there is a function  $f_2$  over  $\mathbf{H}^2 \times S^2$  with  $\alpha_2 = d_2(f_2)$ . One may assume  $f_2 \in L_k^2$ , since  $S^2$  is compact.

There is a function  $g : S^2 \rightarrow \mathbb{R}$  with

$$\alpha_1 = gh_1 + d_1f_1$$

where  $h_1$  is an  $L^2$  harmonic one form over  $\mathbf{H}^2$ , where  $f_1$  also lies in  $L_k^2(\mathbf{H}^2 \times S^2)$ , since  $d_1(\alpha_1) = 0$  vanishes.

It follows from the equality  $d_2\alpha_1 + d_1\alpha_2 = 0$  that the equality holds:

$$d_2g \wedge h_1 + d_2d_1f_1 + d_1d_2f_2 = 0.$$

Then  $d_2g = 0$  holds by algebraic relation, and hence  $g$  is a constant. So  $gh_1$  is an  $L^2$  harmonic one form over  $\mathbf{H}^2 \times S^2$ , and hence can be eliminated from  $\alpha$ .

It follows from the equality  $d\alpha = 0$  that  $d_1d_2(f_1 + f_2) \equiv d_1d_2(f) = 0$  holds. There is a spectral decomposition of  $d_2f = \sum_\lambda a_\lambda u_\lambda$  over  $S^2$ , where  $a_\lambda$  are functions over  $\mathbf{H}^2$ . Then  $d_1a_\lambda = 0$  follows, and hence

$a_\lambda$  are all constants. But since  $d_2 f \in L_{k-1}^2$ , this implies  $d_2 f = 0 = d_2 f_1 + \alpha_2$  holds. So  $\alpha$  has the form  $\alpha = d_1 f - d_2 f$ .

Notice the equalities:

$$\begin{aligned} & \langle d^* d_1(f), g \rangle_{\mathbf{H}^2 \times S^2} = \langle d_1(f), d(g) \rangle = \langle d_1(f), d_1(g) \rangle_{\mathbf{H}^2 \times S^2} \\ & = \int_{S^2} \langle d_1(f), d_1(g) \rangle_{\mathbf{H}^2} = \int_{S^2} \langle d_1^* d_1(f), g \rangle_{\mathbf{H}^2} = \langle d_1^* d_1(f), g \rangle_{\mathbf{H}^2 \times S^2} \end{aligned}$$

Hence  $d^* d_1(f) = d_1^* d_1(f)$  holds. Similar for  $d_2$ . Since  $d^*(\alpha) = 0$  holds, the equality:

$$\Delta_1(f) = \Delta_2(f)$$

must hold. Now since  $\Delta_1$  has continuous spectra only over  $(0, \infty)$ , while  $\Delta_2$  has discrete spectra. This implies the equality  $f = 0$ . This verifies our claim.

**Step 3:** Notice that  $b_+^2(\Sigma_g \times S^2) = 1$  holds, whose element is given by the sum of their volume 2 forms on  $\Sigma_g$  and  $S^2$ . In particular the indices of the AHS complexes over  $\Sigma_g \times S^2$  is  $2 - 2g$ . Then the conclusion follows by the Atiyah's  $\Gamma$  index theorem.  $\square$

**3.4. Some estimates over non compact four manifolds.** In order to apply  $L^p$  estimates over non compact spaces, let us induce some basic inequalities. Let  $M$  be a compact four manifold and  $X = \tilde{M}$  be the universal covering space with  $\Gamma = \pi_1(M)$ .

**Lemma 3.8.** *For  $p \geq 2$ , the global Sobolev embeddings hold:*

$$L_{i+1}^p(X) \subset L_i^{2p}(X)$$

*Proof.* Let  $K \subset X$  be a fundamental domain. Then the local Sobolev estimate gives the embedding  $L_{i+1}^p(K) \subset L_i^{2p}(K)$  in lemma 3.2(1).

Now take  $a \in L_{i+1}^p(X)$ . Then we have the estimate:

$$\|a\|_{L_{i+1}^p(\gamma(K))} \geq c \|a\|_{L_i^{2p}(\gamma(K))}$$

where  $c$  is independent of  $\gamma \in \Gamma$ . So we have the estimates:

$$\begin{aligned} \|a\|_{L_i^{2p}(X)}^{2p} &= \sum_{\gamma \in \Gamma} \|a\|_{L_i^{2p}(\gamma(K))}^{2p} \leq c \sum_{\gamma \in \Gamma} \|a\|_{L_{i+1}^p(\gamma(K))}^{2p} \\ &\leq c \left( \sum_{\gamma \in \Gamma} \|a\|_{L_{i+1}^p(\gamma(K))}^p \right)^2 = c \|a\|_{L_{i+1}^p(X)}^{2p} \end{aligned}$$

$\square$



**Corollary 3.9.** *Let  $p = 2^l \geq 2$ .*

*(1) The embeddings hold:*

$$\mathfrak{H}^{m,p}(X) \supset \mathfrak{H}^{m,2}(X)$$

*between  $L^p$  and  $L^2$  harmonic  $m$ -forms.*

*Proof.* Let  $p = 2^l$ . It follows from lemma 3.8 that the embeddings hold:

$$L_{i+l-1}^2(X) \subset L_{i+l-2}^{2^2}(X) \subset \cdots \subset L_i^p(X)$$

Then the conclusion holds since  $L^p$  harmonic forms have finite Sobolev norms in all  $L_k^p$ . □

**Remark 3.10.** *One wanders the converse embedding. So far such analysis does not seem to be developed so much, even though it is a quite basic subject.*

**3.5.  $L^p$  closedness.** Let us take  $a \in L_k^2(X)$  for some large  $k \gg 1$ . It follows from lemma 3.8 that  $a \in L_1^p(X) \cap L_1^{2p}(X)$  for  $p = 2^l$  with  $l \leq k - 1$ . Suppose the following conditions hold for some constants  $C$  and  $\epsilon_0 > 0$ , and compact subset  $K \subset X$ :

$$(1) \|a\|_{L_1^p(X)} \leq C, \quad (2) \|a\|_{L^p(K)} \geq \epsilon_0, \quad (3) d^*(a) = 0$$

**Lemma 3.11.** *( $\alpha$ ) Suppose the above three conditions (1), (2), (3), and denote  $L^2$  harmonic part by  $a_{\text{harm}} \in \mathfrak{H}^{1,2}$ .*

*Then one of the following criterions hold; the estimates hold:*

$$\|a\|_{L_1^{2p}(X)} \leq c (\|d^+a\|_{L^{2p}(X)} + \|a\|_{\text{harm}})$$

*for some  $c > 0$  independent of  $a$ , or there is a sequence  $\{a_i\}_i$  as above such that they weakly converge in  $L_1^p \cap L_1^{2p}$  to a non zero element in  $\mathfrak{H}^{1,p} \cap \mathfrak{H}^{1,2p}$ , but not in  $\mathfrak{H}^{1,2}$ .*

*( $\beta$ ) Suppose the above condition (3). Then the estimate holds:*

$$\|a\|_{L_1^{2p}(X)} \leq c (\max\{\|d^+a\|_{L^{2p}(X)}, \|a\|_{L_1^p(X)}\} + \|a\|_{\text{harm}}).$$

*Proof.* Let us verify ( $\alpha$ ). Assume that a family  $\{a_i\}_i$  with the above conditions, satisfy the following property:

$$\|a_i\|_{L_1^{2p}(X)} = \delta_i^{-1} (\|d^+(a_i)\|_{L^{2p}(X)} + \|a_i\|_{\text{harm}})$$

for some  $\delta_i \rightarrow 0$ .

Let us divide into two cases;

(a) suppose  $\|a_i\|_{L_1^{2p}(X)}$  are uniformly bounded. Then both convergence  $\|d^+(a_i)\|_{L^{2p}(X)}, \|a_i\|_{\text{harm}} \rightarrow 0$  hold. Then  $\{a_i\}_i$  weakly converge

to some  $a \in L_1^{2p}(X) \cap L_1^p(X)$  with  $d^+(a) = 0$ ,  $a_{\text{harm}} = 0$  and  $d^*(a) = 0$ . Notice that  $L^p$  spaces are reflective for  $1 < p < \infty$ .

Moreover the restriction  $a_i|K$  strongly converge to  $a|K$  in  $L^p$ . It follows from the condition (2) above that  $a$  is non zero and hence gives a non trivial element in  $\mathfrak{H}^{1,p} \cap \mathfrak{H}^{1,2p}$ , but not in  $\mathfrak{H}^{1,2}$ .

(b) Assume  $\|a_i\|_{L_1^{2p}(X)} \rightarrow \infty$ . Then the estimates hold:

$$\begin{aligned} \|a_i\|_{L_1^{2p}(X)} &\leq c(\|d^+(a_i)\|_{L^{2p}(X)} + \|a_i\|_{L^{2p}(X)}) \\ &\leq c\{ \delta_i \|a_i\|_{L_1^{2p}(X)} + \|a_i\|_{L^{2p}(X)} \} \end{aligned}$$

where the first inequality comes from the elliptic estimate. In particular the inequality holds:

$$\|a_i\|_{L_1^{2p}(X)} \leq c' \|a_i\|_{L^{2p}}$$

It follows from lemma 3.8 that the estimate must hold:

$$\|a_i\|_{L_1^{2p}(X)} \leq c' \|a_i\|_{L^{2p}} \leq c'' \|a_i\|_{L_1^p}.$$

Since the left hand side diverges, while  $\|a_i\|_{L_1^p}$  are uniformly bounded by the condition (1). So the case (b) does not happen.

Next consider  $(\beta)$ . If the former conclusion of  $(\alpha)$  holds, then we are done. Otherwise we can take a decreasing sequence  $\delta_i \rightarrow 0$  as in the above proof. Then the same estimates as above give the inequality:

$$\|a_i\|_{L_1^{2p}(X)} \leq c \|a_i\|_{L_1^p(X)}.$$

The conclusion is just combination of these cases. □

**Remark 3.12.** *For our purpose of analysis of the covering monopole map in section 4, any  $p > 2$  case is enough, while  $p = 2$  case is not enough. We will use  $\beta$  only later.*

**3.5.1. Spectral analysis approach.** Let  $D : C_c^\infty(X) \rightarrow C_c^\infty(X)$  be a first order differential operator such that its extension to the Sobolev space  $D : L_k^2(X) \rightarrow L_{k-1}^2(X)$  has closed range.

Let  $p = 2^i$  be power of 2 for  $i \geq 1$ . One considers another extension  $D : L_k^p(X) \rightarrow L_{k-1}^p(X)$  and wants to decide whether they have closed range for  $p \geq 2$ . One possible approach will be to use the spectral decomposition of  $\Delta = D^*D$  over  $L^2$  space. By the assumption, there is a spectral gap near zero so that  $E((0, \delta]) = \phi$  for some positive  $\delta > 0$ , where  $E$  is the spectral density function of  $\Delta$ . Let  $E_1, E_2 \subset L_k^2(X)$  be the closed linear subspaces with  $L_k^2(X) = E_1 \oplus E_2$  such that they are spanned by the vectors in  $E([\delta, \infty))$  and the harmonic  $L^2$  functions respectively.

It follows from lemma 3.8 that there is  $l$  such that  $L_l^p(X)$  is the linear closure of  $L_k^2(X)$  and hence of  $E_1 \oplus E_2$ .  $E_1 \subset L_l^p(X)$  lies in the kernel of  $\Delta$ , and the restriction of  $\Delta$  on  $E_2$  can be expressed by the spectral decomposition:

$$\Delta|_{E_2} = \int_{\delta}^{\infty} \lambda dE_{\lambda}$$

for some positive  $\delta > 0$ . So it amounts to see possibility to extend the functional calculus of the operator on  $L^p$  spaces over  $E_2$ :

$$\int_{\delta}^{\infty} \lambda^{-1} dE_{\lambda}.$$

**3.6. Multiplication estimates.** Let  $X = \tilde{M}$  be the universal covering space of a compact four manifold  $M$ , and  $K \subset X$  be a fundamental domain.

**Lemma 3.13.** *The multiplication:*

$$L_k^2(X) \otimes L_k^2(X) \rightarrow L_m^2(X)$$

*is bounded for  $m \leq k$  with  $k \geq 3$ , or  $m < k$  with  $k \geq 1$ .*

*Proof.* Let us take  $a, b \in L_k^2(X)$  with  $a = \sum_{\gamma \in \Gamma} a_{\gamma}$  with  $a_{\gamma} \in L_k^2(\gamma(K))$ . By corollary 3.3, the local Sobolev multilication gives the estimates:

$$\|a_{\gamma} b_{\gamma}\|_{L_m^2} \leq C \|a_{\gamma}\|_{L_k^2} \|b_{\gamma}\|_{L_k^2}$$

where  $C$  is independent of  $\gamma \in \pi_1(M)$ . So

$$\begin{aligned} \|ab\|_{L_m^2(X)}^2 &= \sum_{\gamma} \|a_{\gamma} b_{\gamma}\|_{L_m^2}^2 \\ &\leq C \sum_{\gamma} \|a_{\gamma}\|_{L_k^2}^2 \|b_{\gamma}\|_{L_k^2}^2 \leq C \left( \sum_{\gamma} \|a_{\gamma}\|_{L_k^2}^2 \right) \left( \sum_{\gamma} \|b_{\gamma}\|_{L_k^2}^2 \right) \\ &= \|a\|_{L_k^2(X)}^2 \|b\|_{L_k^2(X)}^2. \end{aligned}$$

□

**Lemma 3.14.** *Let  $m > 2$ . If two elements  $a, b \in L_{2m}^2(X)$  with  $\|a\|_{L_{2m}^2(X)} = \|b\|_{L_{2m}^2(X)} = 1$  satisfy uniformly small estimates:*

$$\|a\|_{L_{2m}^2(\gamma K)}, \|b\|_{L_{2m}^2(\gamma K)} < \epsilon$$

*for all  $\gamma \in \Gamma$ , then there is some constant  $C = C_K$  independent of  $\epsilon > 0$  such that the estimate:*

$$\|ab\|_{L_{2m}^2(X)} < C\epsilon$$

*holds.*

*Proof.* It follows from the local Sobolev embedding  $C^m \hookrightarrow L_{2m}^2$  in lemma 3.2(2) that the estimates:

$$\|a\|_{C^m(X)}, \|b\|_{C^m(X)} < C\epsilon$$

hold.

Consider the absolute values of the derivatives for  $l \leq 2m$ :

$$|\nabla^l(ab)| \leq \sum_{s=0}^l |\nabla^s(a)| |\nabla^{l-s}(b)|$$

where each component in the right hand side satisfies the property that one of  $s$  or  $l-s$  is less than or equal to  $m$ . Suppose  $s \leq m$  holds. Then we have the estimates:

$$\begin{aligned} |\nabla^s(a)|^2 |\nabla^{l-s}(b)|^2 &\leq C^2 \epsilon^2 |\nabla^{l-s}(b)|^2 \\ &\leq C^2 \epsilon^2 (|\nabla^s(a)|^2 + |\nabla^{l-s}(b)|^2) \end{aligned}$$

We can obtain the same estimate when  $l-s \leq m$  holds by the same argument.

So in any case the estimates hold:

$$|\nabla^l(ab)|^2 \leq C' \epsilon^2 \sum_{s=0}^l (|\nabla^s(a)|^2 + |\nabla^s(b)|^2)$$

Now we obtain the estimate by integration:

$$\|ab\|_{L_{2m}^2(X)} \leq C\epsilon(\|a\|_{L_{2m}^2(X)} + \|b\|_{L_{2m}^2(X)}).$$

□

**Corollary 3.15.** *Take two elements  $b \in L_m^2(X)$  and  $a \in L_{2m}^2(X)$ , and assume  $a$  satisfies the uniformly small estimate:*

$$\|a\|_{L_{2m}^2(\gamma K)} < \epsilon$$

*for all  $\gamma \in \Gamma$ . Then there is some constant  $C = C_K$  such that the estimate:*

$$\|ab\|_{L_m^2(X)} < C\epsilon\|b\|_{L_m^2(X)}$$

*holds.*

*Proof.* Consider the absolute values of the derivatives for  $l \leq m$ :

$$|\nabla^l(ab)| \leq \sum_{s=0}^l |\nabla^s(a)| |\nabla^{l-s}(b)|.$$

Then the same argument as above gives the estimate:

$$|\nabla^l(ab)|^2 \leq C' \epsilon^2 \sum_{s=0}^l |\nabla^s(b)|^2$$

Hence we obtain the estimates:

$$\|ab\|_{L_m^2(X)} \leq C\epsilon\|b\|_{L_m^2(X)}$$

□

**Remark 3.16.** Let  $K \subset X$  be a fundamental domain, and choose a finite set  $\bar{\gamma} \equiv \{\gamma_1, \dots, \gamma_m\} \subset \pi_1(M) = \Gamma$ . For positive constant  $\epsilon > 0$ , let us put:

$$H'(\epsilon, \bar{\gamma}) \equiv \{ w \in L_k^2(X; E) = H' : \|w\|_{L_k^2(\gamma K)} < \epsilon, \gamma \notin \bar{\gamma} \}.$$

(1) For any  $r$  and  $r$  ball  $D_r \subset H'$ , there is some  $m$  such that the embedding  $D_r \subset H'(\epsilon, m) \equiv \cup_{\bar{\gamma} \in \Gamma^m} H'(\epsilon, \bar{\gamma})$  holds.

(2) By lemma 3.14, there is a constant  $C$  such that the covering Seiberg-Witten map restricts as:

$$F : H'(\epsilon, \bar{\gamma}) \rightarrow H(C\epsilon, \bar{\gamma}).$$

**3.7. Locality of linear operators.** Let  $K \subset X$  be a compact subset. Suppose  $l : L_k^2(X) \rightarrow L_{k-1}^2(X)$  is a differential operator of first order, and consider its restriction:

$$l : L_k^2(K)_0 \rightarrow L_{k-1}^2(K)_0$$

between the Sobolev spaces over a compact subset  $K$ . Let us take an element  $w \in l(L_k^2(X)) \cap L_{k-1}^2(K)_0$ , and ask when  $w$  lies in the image  $l(L_k^2(K)_0)$ . In general it is not always the case. Later when we consider properness of the covering monopole map, we shall use projection to the Sobolev spaces over compact subsets. Here let us observe a general analytic property.

Let us introduce an *error quantity*  $e(w) \in [0, \infty)$  as:

$$e(w; K) = \{ \|v\|_{L_k^2(K^c)} : l(v) = w \}.$$

Let  $K_0 \subset\subset K_1 \subset\subset K_2 \subset \dots \subset X$  be an exhaustion.

**Lemma 3.17.** Suppose  $l : L_k^2(X) \rightarrow L_{k-1}^2(X)$  is injective with closed range. Choose  $w_i \in L_{k-1}^2(K_i)_0 \cap l(L_k^2(X))$  which converge to  $w \in L_{k-1}^2(X)$ .

Then the error quantities go to zero:

$$e(w_i; K_i) \rightarrow 0.$$

*Proof.* By the assumption, there are  $v_i \in L_k^2(X)$  which converge to  $v \in L_k^2(X)$  with  $l(v_i) = w_i$  and  $l(v) = w$ . For small  $\epsilon > 0$ , there is  $i_0$  such that  $\|w - w_i\|_{L_{k-1}^2(X)} < \epsilon$  hold for  $i \geq i_0$ . It follows from the assumption on  $l$  that the estimates  $\|v - v_i\|_{L_k^2(X)} \leq C\epsilon$  hold for a constant  $C$ .

Suppose there could exist  $\delta > 0$  with  $e(w_i; K_i) \geq \delta$ . Then we have the estimates:

$$\|v\|_{L_k^2(K_i^c)} \geq \|v_i\|_{L_k^2(K_i^c)} - \|v - v_i\|_{L_k^2(K_i^c)} \geq \delta - C\epsilon > 0$$

for all sufficiently large  $i$ , which cannot happen. □

**3.8. More Sobolev estimates.** Here we verify the Sobolev estimate, which improves the original version to a most general way. This is due to Furuta. The estimate will not be used in later sections, even though it seemed unknown so far.

**Lemma 3.18.** *Suppose (1)  $k - \frac{4}{p} \geq l - \frac{4}{q}$  with  $k \geq l$ , and (2)  $p \leq q$ . Then the embeddings hold:*

$$L_k^p(X) \subset L_l^q(X)$$

over the universal covering space  $X = \tilde{M}$  of a compact 4 manifold.

*Proof.* It follows from the assumption (1) that the local Sobolev embedding  $(L_k^p)_{\text{loc}} \subset (L_l^q)_{\text{loc}}$  holds by lemma 3.2(1).

Take  $a \in L_k^p(X)$ . Then we obtain the estimate:

$$(\|a\|_{L_k^p(X)})^q = (\sum_{\gamma \in \Gamma} \|a\|_{L_k^p(\gamma(K))}^p)^{\frac{q}{p}} \geq c(\sum_{\gamma \in \Gamma} \|a\|_{L_l^q(\gamma(K))}^p)^{\frac{q}{p}}$$

We want to verify the inequality:

$$(\sum_{\gamma \in \Gamma} \|a\|_{L_l^q(\gamma(K))}^p)^{\frac{q}{p}} \geq \sum_{\gamma \in \Gamma} \|a\|_{L_l^q(\gamma(K))}^q$$

The following sub lemma finishes the proof of lemma:

**Sublemma 3.19.** *Let  $\{a_i\}_{i=0}^\infty$  be a non negative sequence. Then the estimates:*

$$\sum_i a_i \leq (\sum_i a_i^{t^{-1}})^t$$

hold for  $t \geq 1$ .

*Proof.* It is enough to verify the estimates  $\sum_{i=0}^N a_i \leq (\sum_{i=0}^N a_i^{t^{-1}})^t$  for all  $N$ .

Let  $N = 2$ , and verify the estimate:

$$a_0 + a_1 \leq (a_0^{t^{-1}} + a_1^{t^{-1}})^t.$$

One may assume  $a_0 \leq a_1$ , and then may assume  $a_0 = 1$  by multiplying  $a_0^{-1}$  on both sides. Then the estimate  $1 + a_1 \leq (1 + a_1^{t^{-1}})^t$  for  $a_1 \geq 1$  is rewritten as  $1 + a^t \leq (1 + a)^t$  for  $a = a_1^{t^{-1}} \geq 1$  and  $t \geq 1$ . It certainly holds when  $t$  is positive integer by the Taylor expansion. Suppose  $t = \frac{p}{q}$  with positive integers  $p, q$ , and put  $b = a^{q^{-1}} \geq 1$ . Then the estimate is given by  $(1 + b^q)^q \leq (1 + b^p)^p$ .  $f(s) = (1 + b^s)^s$  is monotone increasing function, since  $(\log f)(s) = s \log(1 + b^s)$  is the case for  $b \geq 1$  and  $s \geq 1$ .

In particular the inequality  $1 + a^t \leq (1 + a)^t$  hold for all  $t \in [1, \infty) \cap \mathbb{Q}$ . By continuity, the same estimate holds for all  $t \in [1, \infty)$ .

Suppose the estimate hold up to  $N - 1$ . Then:

$$\sum_{i=0}^N a_i \leq \left( \sum_{i=0}^{N-1} a_i^{t^{-1}} \right)^t + a_N \leq \left( \sum_{i=0}^{N-1} a_i^{t^{-1}} + a_N^{t^{-1}} \right)^t = \left( \sum_{i=0}^N a_i^{t^{-1}} \right)^t$$

So the induction step finishes.  $\square$

$\square$

#### 4. PROPERNESS OF THE MONOPOLE MAP

A metrically proper map between Hilbert spaces satisfies the property that pre-image of a bounded set is also bounded. In order to obtain a stronger property of the map for our purpose, we need another condition that the restriction on any bounded set is locally proper. It is the case for the monopole map over a compact manifold, since it is Fredholm. In our case the base space is non compact, and the covering monopole map satisfies such property locally in our sense. We will verify local properness under the assumption of closeness of the AHS complex.

Let us fix a solution  $(A_0, \psi_0)$  to the Seiberg-Witten equation over  $M$  and choose the pair as the base point of the covering monopole map. In this section, we shall verify the following property:

**Theorem 4.1.** *Suppose the AHS complex is closed over  $X = \tilde{M}$ .*

*Then it is locally strongly proper so that it is strongly proper over each compact subset  $K \subset X$ :*

$$\begin{aligned} \tilde{\mu} : L_k^2(K; \tilde{S}^+)_0 \oplus L_k^2(X; \Lambda^1) \cap \text{Ker } d^* \\ \rightarrow L_{k-1}^2(K; \tilde{S}^-)_0 \oplus L_{k-1}^2(X; \Lambda_+^2) \oplus H^1(X) \\ (\phi, a) \rightarrow (D_{\tilde{A}_0+a} \phi, d^+(a) - \sigma(\tilde{\psi}_0, \phi), [a]). \end{aligned}$$

*In particular the conclusion holds if the linearized map of the covering monopole map gives an isomorphism.*

*Proof.* The conclusion follows from combination of lemma 4.7 and proposition 4.9 with lemma 4.10 which is a version of lemma 4.4. These are all verified later in this chapter.  $\square$

We shall verify global properness for a particular class, which is stronger than locally strong properness.

**Proposition 4.2.** *Suppose the AHS complex has closed range over  $X = \tilde{M}$  whose second  $L^2$  cohomology  $H_{AHS}^2(X) = 0$  vanishes. If the metric on  $M$  has positive scalar curvature, then the covering monopole map is strongly proper.*

We shall present examples which satisfy the above conditions. Actually  $S^2 \times \Sigma_g$  are the cases for all  $g \geq 2$ .

Our strategy is the following. Assume the AHS complex is closed. Then we verify:

- (1) metrical properness for  $k = 1$  under the assumption of existence of positive scalar metric (lemma 4.4)
- (2) locally metrical properness for  $k = 1$  (lemma 4.10)
- (3) (locally) metrical properness for  $k \geq 1$  under the additional two assumptions of (local)  $L^\infty$  bound and (locally) metrical properness for  $k = 1$  (lemma 4.7)
- (4) local  $L^\infty$  bounds (proposition 4.9)

**Remark 4.3.** *Let  $D \subset M$  be a small ball. There exists a Riemannian metric  $g$  such that its scalar curvature is positive except  $D$  [KW]. One may assume that the lift  $\tilde{D} \subset X$  satisfies that  $\gamma(\tilde{D}) \cap \tilde{D} = \emptyset$  for all  $\gamma \neq id$ . Let us put  $\bar{D} \equiv \cup_{\gamma \in \Gamma} \gamma(\tilde{D})$ .*

*Assume that  $\tilde{\mu}$  could be metrically non proper, and choose  $\tilde{\mu}(x_i) = y_i$  such that  $\|y_i\| \leq c < \infty$  while  $\|x_i\| \rightarrow \infty$ . Let  $\varphi$  be a cut off function with  $\varphi|_{\bar{D}} \equiv 1$  and zero outside a small neighborhood of  $\bar{D}$ . Both two families  $\tilde{\mu}|_{\{(1-\varphi)x_i\}_i}$  and  $\tilde{\mu}|_{\{\varphi x_i\}_i}$  must be proper (see lemma 4.4 below and 2.2), and hence  $\{x_i\}_i$  should be unbounded near the boundary of some  $\gamma(\bar{D})$  and  $\gamma \in \Gamma$ .*

**4.1. Positive scalar curvature metric.** Let us verify metrical properness of the monopole map, in the case when (1) the base manifold  $M$  admits a positive scalar metric and (2) the AHS complex over  $X = \tilde{M}$  is closed.

Let us fix a reducible solution  $(A_0, 0)$  to the Seiberg-Witten equation over  $M$  and choose the pair as the base point of the covering monopole map.

**Lemma 4.4.** *Suppose the AHS complex has closed range over  $X = \tilde{M}$ . If  $M$  admits a Riemannian metric of positive scalar curvature, then the*



covering monopole map:

$$\begin{aligned}\tilde{\mu} : L_1^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* \rightarrow \\ L^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) \\ (\phi, a) \rightarrow (F_{\tilde{A}_0, 0}(\phi, a), [a])\end{aligned}$$

is metrically proper, where  $H^1(X)$  is the first  $L^2$  cohomology group.

*Proof.* Let us put  $\tilde{\mu}(\phi, a) = (\varphi, b, h)$ , and denote  $A = \tilde{A}_0 + a$ .

**Step 1:** We have the pointwise equalities:

$$\begin{aligned}(F_A \phi, \phi) &= (F_A^+ \phi, \phi) = ((F_A^+ - \sigma(\phi))\phi + \sigma(\phi)\phi, \phi) \\ &= (b\phi, \phi) + \frac{|\phi|^4}{2}\end{aligned}$$

since  $F_A \phi = F_A^+ \phi$  holds.

Suppose  $M$  admits a Riemannian metric of positive scalar curvature. Then it follows from the Weitzenbock formula:

$$D_A^2(\phi) = \nabla_A^* \nabla_A(\phi) + \frac{\kappa}{4}\phi + \frac{F_A}{2}\phi$$

that the estimates hold:

$$\|D_A(\phi)\|_{L^2}^2 + \|b\|_{L^2}\|\phi\|_{L^4} \geq \delta\|\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^4}^4 \geq \frac{1}{4}\|\phi\|_{L^4}^4$$

for some positive  $\delta > 0$ . In particular there is  $c = c(\|\varphi\|_{L^2}, \|b\|_{L^2})$  such that the bound:

$$\|\phi\|_{L^4} \leq c$$

holds. By use of another estimate:

$$\|\varphi\|_{L^2}^2 + \|b\|_{L^2}\|\phi\|_{L^4} \geq \delta\|\phi\|_{L^2}^2 + \frac{1}{4}\|\phi\|_{L^4}^4 \geq \delta\|\phi\|_{L^2}^2$$

we obtain  $L^2$  estimate:

$$\|\phi\|_{L^2} \leq c'(\|\varphi\|_{L^2}, \|b\|_{L^2}, \delta)$$

**Step 2:** It follows from the equality  $d^+(a) = b + \sigma(\phi)$  that we have the estimates:

$$\begin{aligned}\|a\|_{L_1^2} &\leq c(\|d^+(a)\|_{L^2} + \|a_{\text{harm}}\|_{L^2}) \\ &\leq c(\|b\|_{L^2} + \|\phi\|_{L^4} + \|h\|_{L^2})\end{aligned}$$

Combinig with step 1, we have the estimate:

$$\|a\|_{L_1^2} \leq c(\|\varphi\|_{L^2}, \|b\|_{L^2}, \|h\|_{L^2}, \delta)$$

**Step 3:** It follows from the embedding  $L_1^2 \subset L^4$  in lemma 3.8 that we have the estimates:

$$\|a\phi\|_{L^2} \leq \|a\|_{L^4}\|\phi\|_{L^4} < \infty$$

Then we have the estimates:

$$\|D_{\tilde{A}_0}(\phi)\|_{L^2} \leq \|D_A(\phi)\|_{L^2} + \|a\phi\|_{L^2} < \infty$$

It follows from the elliptic estimate that we have the bounds of  $\phi$  in  $L_1^2$ :

$$\|\phi\|_{L_1^2} \leq c(\|\varphi\|_{L^2}, \|b\|_{L^2}, \|h\|_{L^2}, \delta)$$

□

**Remark 4.5.** *In order to induce higher Sobolev estimates, it is enough to obtain the estimate of  $\|a\phi\|_{L_1^2}$ . We can obtain the immediate estimate*

$$\|a\phi\|_{L_k^2} \leq C_k \|a\|_{L_k^2} \|\phi\|_{L_k^2}$$

for  $k \geq 3$ , but it is not applied at  $k = 1$ . This forces us to use  $L^\infty$  estimates later, which leads to  $L^p$  analysis.

**Example 4.6.** *An immediate examples of closed four manifolds with positive scalar curvature will be  $S^4$  or  $\Sigma_g \times S^2$  with their metrics  $h + \epsilon g$  with small  $\epsilon > 0$ , where  $h$  and  $g$  are both the standard metrics. For the latter case the AHS complex over their universal covering spaces have closed range by lemma 3.6.*

**4.2. Regularity under  $L^\infty$  bounds.** Let us take a solution  $(A_0, \psi_0)$  to the Seiberg-Witten equation over  $M$ , and consider the covering monopole map with the base  $(A_0, \psi_0)$ .

**Lemma 4.7.** *Assume the AHS complex has closed range.*

*Suppose the monopole map:*

$$\begin{aligned} \tilde{\mu} : L_1^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* &\rightarrow \\ L^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) & \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\phi, a), [a]) \end{aligned}$$

*is metrically proper. If moreover  $L^\infty$  estimates are given:*

$$\|(\phi, a)\|_{L^\infty} \leq c < \infty$$

*Then the monopole map:*

$$\begin{aligned} \tilde{\mu} : L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* &\rightarrow \\ L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) & \\ (\phi, a) &\rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\phi, a), [a]) \end{aligned}$$

*is metrically proper.*

*Proof. Step 1:* Let us take  $(\phi, a) \in L_k^2((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^*$ , and put  $\tilde{\mu}(\phi, a) = (\varphi, b, h)$ . It follows from the assumption that both  $\|(\phi, a)\|_{L^\infty}$  and  $\|(\phi, a)\|_{L_1^2}$  are bounded by a constant  $C = C(r)$  with  $r = \|(\varphi, b, h)\|_{L_{k-1}^2}$ .

**Step 2:** We claim that:

$$a\phi \in L_1^2(X)$$

holds and bounded by a constant  $C = C(r)$ .

In fact firstly  $a\phi \in L^2(X)$  holds by  $L^\infty$  bounds:

$$\|a\phi\|_{L^2} \leq \|a\|_{L^\infty} \|\phi\|_{L^2} \leq C(r)$$

Notice the equality  $\nabla(a\phi) = \nabla(a)\phi + a\nabla(\phi)$ . Hence

$$\|\nabla(a\phi)\|_{L^2} \leq \|\phi\|_{L^\infty} \|\nabla(a)\|_{L^2} + \|a\|_{L^\infty} \|\nabla(\phi)\|_{L^2} \leq C(r).$$

This verifies the claim.

One may assume that the base solution  $[A_0, \psi_0]$  is smooth on  $M$ . So  $\|a\tilde{\psi}_0\|_{L_1^2} \leq C\|a\|_{L_1^2}$  is bounded. Then since the left hand side of:

$$\varphi = D_A(\phi) + a\tilde{\psi}_0 = D_{\tilde{A}_0}(\phi) + a\phi + a\tilde{\psi}_0$$

has  $L_{k-1}^2(X)$  norm less than  $r$ , it follows  $D_{\tilde{A}_0}(\phi) \in L_1^2(X)$  and hence  $\phi \in L_2^2(X)$  by the elliptic estimate and the assumption.

Then  $\phi \in L_1^4(X)$  holds, since the embedding  $L_2^2(X) \subset L_1^4(X)$  holds by lemma 3.8.

Let us denote  $\sigma(\tilde{\psi}_0, \phi) = \sigma(\phi) + l(\tilde{\psi}_0, \phi)$  (see lemma 2.2). We obtain the estimates:

$$\|\nabla\sigma(\phi)\|_{L^2} \leq \|\nabla(\phi)\|_{L^4} \|\phi\|_{L^4} \leq \|\phi\|_{L_1^4} \|\phi\|_{L^4} \leq C(r),$$

$$\|\nabla l(\tilde{\psi}_0, \phi)\|_{L^2} \leq C(\|\phi\|_{L^2} + \|\nabla\phi\|_{L^2}) \leq C\|\phi\|_{L_1^2} \leq C(r).$$

Since  $b = d^+(a) - \sigma(\tilde{\psi}_0, \phi)$  has  $L_{k-1}^2(X)$  norm less than  $r$ , we have obtained the bound  $\|d^+(a)\|_{L_1^2(X)} \leq C(r)$ . Then  $\|a\|_{L_2^2} \leq C(r)$  follows from closedness of the AHS complex.

**Step 3:** Let us verify that  $a\phi \in L_2^2(X)$  holds and bounded by a constant  $C = C(r)$ . Notice the equality

$$\nabla^2(a\phi) = \nabla^2(a)\phi + 2\nabla(a)\nabla(\phi) + a\nabla^2(\phi)$$

The first and last terms have both bounded  $L^2(X)$  norms by  $L^\infty$  bounds of  $(a, \phi)$ . For the middle one, we have the estimates:

$$\begin{aligned} \|\nabla(a)\nabla(\phi)\|_{L^2} &\leq \|\nabla(a)\|_{L^4} \|\nabla(\phi)\|_{L^4} \leq C\|\nabla(a)\|_{L_1^2} \|\nabla(\phi)\|_{L_1^2} \\ &\leq C\|a\|_{L_2^2} \|\phi\|_{L_2^2} \end{aligned}$$

where the last inequality holds by lemma 3.8. This verifies the inclusion  $a\phi \in L_2^2(X)$ .

The rest argument is parallel to step 2.  $D_{\tilde{A}_0}(\phi) \in L_2^2(X)$  and hence  $\phi \in L_3^2(X)$ . Since  $\sigma(\tilde{\psi}_0, \phi) \in L_2^2(X)$  holds, we obtain  $d^+(a) \in L_2^2(X)$ . Then we have  $a \in L_3^2(X)$  by closedness of the AHS complex. All their norms are bounded by constants  $C(r)$ .

**Step 4:** We have verified  $a, \phi \in L_3^2(X)$ . Now we can use somehow simpler argument that the multiplication:

$$L_3^2(X) \times L_3^2(X) \rightarrow L_3^2(X)$$

is continuous in lemma 3.13 so that we can see the inclusion  $a\phi \in L_3^2(X)$  immediately:

$$\|a\phi\|_{L_3^2(X)}^2 \leq C\|a\|_{L_3^2(X)}^2\|\phi\|_{L_3^2(X)}^2 \leq C(r).$$

Then we repeat the latter part of step 2, and obtain the inclusions  $\phi \in L_4^2(X)$  and  $a \in L_4^2(X)$ .

The rest argument is parallel, and we obtain  $L_k^2(X)$  bound of  $(a, \phi)$  by a constant  $C(r)$ .

□

**Remark 4.8.** *The above proof also verifies that one can restrict the functional spaces to  $L_1^2(K; \tilde{S}^+)_0 \oplus L_1^2(X; \Lambda^1 \otimes i\mathbb{R})_0 \cap \text{Ker } d^*$ , and still obtains the same conclusion so that locally metrical properness holds.*

**4.3.  $L^\infty$  estimates.** Let us take a solution  $(A_0, \psi_0)$  to the Seiberg-Witten equation over  $M$ , and consider the covering monopole map with the base  $(A_0, \psi_0)$ .

Let us take an element:

$$(\phi, a) \in L_k^2((K', g); \tilde{S}^+) \oplus L_k^2((X, g); \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^*$$

and put  $\tilde{\mu}(\phi, a) = (\varphi, b, h)$  and  $r = \|(\varphi, b, h)\|_{L_{k-1}^2}$ , where  $K' \subset X$  is a compact subset.

**Proposition 4.9.** *Suppose that the AHS complex has closed range.*

*Then we have  $L^\infty$  estimates:*

$$(\phi, a) \in L^\infty$$

*in terms of  $r = \|(\varphi, b, h)\|_{L_{k-1}^2}$  and  $K'$ .*

*Proof.* We verify the conclusion when the base solution is reducible  $(A_0, 0)$  at step 2. The general case is verified at step 3.

**Step 1:** We claim that the uniform estimate holds:

$$\|a\|_{L_1^8} \leq C(\|d^+(a_i)\|_{L^p(X)} + r)$$

for at least one of  $p = 2, 4$  or  $8$ .

It follows from lemma 3.11( $\beta$ ) that the inequality holds:

$$\|a\|_{L_1^8(X)} \leq c (\max\{\|d^+a\|_{L^8(X)}, \|a_i\|_{L_1^4(X)}\} + \|a\|_{harm}).$$

By the same lemma again, we have another inequality:

$$\begin{aligned} \|a\|_{L_1^4(X)} &\leq c (\max\{\|d^+a\|_{L^4(X)}, \|a\|_{L_1^2(X)}\} + \|a\|_{harm}) \\ &\leq C(\max\{\|d^+a\|_{L^4(X)}, \|d^+(a)\|_{L^2}\} + \|a\|_{harm}) \end{aligned}$$

where we have used the assumption of closeness of AHS complex for  $k = 1$  at the second inequality. So we have verified the claim by combination of these two estimates.

**Step 2:** It follows from step 1 with the Sobolev estimate that the uniform estimate holds:

$$\|a\|_{L^\infty} \leq c\|a\|_{L_1^8} \leq C(\|d^+(a_i)\|_{L^p(X)} + \|a_{harm}\| + r)$$

for at least one of  $p = 2, 4$  or  $8$ .

The Weitzenböck formula gives the equality:

$$D_A^* D_A = \nabla_A^* \nabla_A + \frac{1}{4}s + \frac{1}{2}F_A^+$$

where  $s$  is the scalar curvature.

Now suppose the base point is reducible  $(A_0, 0)$  over  $M$ . Let  $\Delta$  be the Laplacian on functions. We have the point wise estimate (see [BF] p12):

$$\Delta|\phi|^2 \leq < 2D_A^* \varphi - \frac{s}{2}\phi - (b + \sigma(\phi))\phi, \phi > .$$

Then the point wise estimate:

$$\begin{aligned} \Delta|\phi|^2 + \frac{s}{2}|\phi|^2 + \frac{1}{2}|\phi|^4 &= < 2D_{A_0}^* \varphi, \phi > + < 2a\varphi, \phi > - < b\phi, \phi > \\ &\leq 2(\|D_{A_0}^* \varphi\|_{L^\infty} + \|a\|_{L^\infty}\|\varphi\|_{L^\infty})|\phi| + \|b\|_{L^\infty}|\phi|^2 \end{aligned}$$

holds by use of the equality  $\sigma(\phi)\phi = \frac{|\phi|^2}{2}\phi$ .

By the assumption, there is a compact subset  $K' \subset X$  so that  $\phi$  has compact support inside  $K'$ . Notice a priori estimate:

$$\|\phi\|_{L^p(X)} = \|\phi\|_{L^p(K')} \leq C\|\phi\|_{L^\infty(K')} = C\|\phi\|_{L^\infty(X)}$$

for some constant  $C = C_{K'}$ . Then combining with the equality  $d^+a = b + \sigma(\phi)$ , we have the estimates:

$$\begin{aligned} \|a\|_{L^\infty} &\leq c(\|a_{harm}\| + \|b\|_{L^p} + \|\phi\|_{L^\infty}^2 + r) \\ &\leq c(\|a_{harm}\| + \|b\|_{L_2^2} + \|\phi\|_{L^\infty}^2 + r) \\ &\leq c(\|a_{harm}\| + \|b\|_{L_{k-1}^2} + \|\phi\|_{L^\infty}^2 + r) \end{aligned}$$

by lemma 3.8.

For  $\varphi$ , we have the estimates by lemma 3.8:

$$\|\varphi\|_{L^\infty} \leq C\|\varphi\|_{L_1^8(X)} \leq C'\|\varphi\|_{L_3^2(X)} \leq C'\|\varphi\|_{L_{k-1}^2(X)}.$$

At the maximum of  $|\phi|^2$ ,  $\Delta|\phi|^2$  is non negative, and hence we obtain the estimate:

$$\|\phi\|_{L^\infty}^4 \leq c(\|a_{harm}\|, \|b\|_{L_{k-1}^2}, \|\varphi\|_{L_{k-1}^2}, r)(\|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2 + \|\phi\|_{L^\infty}^3).$$

So we have  $L^\infty$  estimates of the pair  $(\phi, a)$  by  $(\|\varphi\|_{L_{k-1}^2}, \|b\|_{L_{k-1}^2}, \|h\|)$ .

**Step 3:** Let us induce  $L^\infty$  bound for the case of general base  $[A_0, \psi_0]$ . We will follow step 1 and 2.

It follows from step 1 that the uniform estimates hold:

$$\|a\|_{L^\infty} \leq c\|a\|_{L_1^8} \leq c(\|d^+a\|_{L^p} + \|a_{harm}\| + r)$$

for at least one of  $p = 2, 4$  or  $8$ .

Let us put  $\phi_0 \equiv \tilde{\psi}_0 + \phi$ . Notice the bound:

$$-c + \|\phi\|_{L^\infty} \leq \|\phi_0\|_{L^\infty} \leq c + \|\phi\|_{L^\infty}$$

holds since  $\tilde{\psi}_0$  is  $\Gamma$  invariant. The Weitzenböck formula gives the point wise estimate:

$$\begin{aligned} \Delta|\phi_0|^2 &\leq \langle 2D_A^*D_A\phi_0 - \frac{s}{2}\phi_0 - (b + \sigma(\phi_0))\phi_0, \phi_0 \rangle \\ &= \langle 2D_A^*(\varphi) - \frac{s}{2}\phi_0 - (b + \sigma(\phi_0))\phi_0, \phi_0 \rangle \end{aligned}$$

Now we have the estimates:

$$\begin{aligned} \Delta|\phi_0|^2 &+ \frac{s}{2}|\phi_0|^2 + \frac{|\phi_0|^4}{2} \\ &\leq 2|D_A(\varphi)| |\phi_0| + \langle b\phi_0, \phi_0 \rangle \\ &\leq 2(\|D_{\tilde{A}_0}(\varphi)\|_{L^\infty} + \|a\|_{L^\infty}\|\varphi\|_{L^\infty}) |\phi_0| + \|b\|_{L^\infty} |\phi_0|^2 \\ &\leq 2(\|D_{\tilde{A}_0}(\varphi)\|_{L_3^2} + \|a\|_{L^\infty}r) |\phi_0| + r |\phi_0|^2 \\ &\leq 2r(c + \|a\|_{L^\infty}) |\phi_0| + r |\phi_0|^2 \end{aligned}$$

where we have used the estimates  $\|b\|_{L^\infty} \leq c\|b\|_{L_1^8} \leq c\|b\|_{L_3^2} \leq r$ , and the elliptic estimate.

By the assumption, there is a compact subset  $K' \subset X$  and a constant  $C = C_{K'}$  so that  $\phi$  has compact support inside  $K'$ . Notice the estimates  $\|\phi\|_{L^p(X)} \leq C\|\phi\|_{L^\infty(X)}$ , and:

$$\|\sigma(\tilde{\psi}_0, \phi)\|_{L^p} \leq C(\|\phi\|_{L^p} + \|\phi\|_{L^{2p}}^2) \leq C(\|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2)$$

Then combining step 1 with the equality  $d^+a = b + \sigma(\tilde{\psi}_0, \phi)$ , we have the estimates:

$$\begin{aligned}
\|a\|_{L^\infty} &\leq c(\|b\|_{L^p} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2 + r) \\
&\leq c(\|b\|_{L^2_2} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2 + r) \\
&\leq c(\|b\|_{L^2_{k-1}} + \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2 + r) \\
&\leq c'(\|\phi\|_{L^\infty} + \|\phi\|_{L^\infty}^2 + r) \\
&\leq c'(\|\phi_0\|_{L^\infty} + \|\phi_0\|_{L^\infty}^2 + r + 1)
\end{aligned}$$

Let us combine the above estimates. At the maximum of  $|\phi_0|^2$ ,  $\Delta|\phi_0|^2$  is non negative, and hence we obtain the estimates:

$$\begin{aligned}
\|\phi_0\|_{L^\infty}^4 &\leq 4r\{(c + \|a\|_{L^\infty})\|\phi_0\|_{L^\infty} + \|\phi_0\|_{L^\infty}^2\} \\
&= 4r(c + \|\phi_0\|_{L^\infty})\|\phi_0\|_{L^\infty} + \|a\|_{L^\infty}\|\phi_0\|_{L^\infty} \\
&\leq \{4r(c + \|\phi_0\|_{L^\infty}) + c'(\|\phi_0\|_{L^\infty} + \|\phi_0\|_{L^\infty}^2 + r + 1)\}\|\phi_0\|_{L^\infty} \\
&\leq \{4r\|\phi_0\|_{L^\infty} + c'(\|\phi_0\|_{L^\infty} + \|\phi_0\|_{L^\infty}^2 + r + 1)\}\|\phi_0\|_{L^\infty}.
\end{aligned}$$

So we obtain  $L^\infty$  estimate of the pair  $(\phi, a)$  by  $(\|\varphi\|_{L^2_{k-1}}, \|b\|_{L^2_{k-1}}, \|h\|)$ .  $\square$

#### 4.4. $L^1$ bounds.

**Lemma 4.10.** *Let  $(A_0, \psi_0)$  be a solution to the Seiberg-Witten equation over  $M$  and  $\tilde{\mu}$  be the covering monopole map with the base  $(A_0, \psi_0)$ . If the AHS complex has closed range over  $X = \tilde{M}$ , then for a compact subset  $K \subset X$ , the restriction of the monopole map:*

$$\begin{aligned}
\tilde{\mu} : L^2_1(K; \tilde{S}^+)_0 \oplus L^2_1(X; \Lambda^1) \cap \text{Ker } d^* \\
\rightarrow L^2(K; \tilde{S}^-)_0 \oplus L^2(X; \Lambda^2_+; X) \oplus H^1(X) \\
(\phi, a) \rightarrow (D_{\tilde{A}_0, \tilde{\psi}_0}(a, \phi), d^+(a) - \sigma(\tilde{\psi}_0, \phi), [a])
\end{aligned}$$

is metrically proper, where  $H^1(X)$  is the first  $L^2$  cohomology group.

*Proof.* We follow the argument in lemma 4.4. Let us denote  $\tilde{\mu}(\phi, a) = (\varphi, b, h)$  with  $\phi \in L^2_1(K)_0$ .

Notice the local Sobolev estimates:

$$\|\phi\|_{L^2(K)_0} \leq C_K \|\phi\|_{L^3(K)_0} \leq C'_K \|\phi\|_{L^4(K)_0}$$

**Step 1:** Firstly suppose the base point is reducible  $(A_0, 0)$ . Following step 1 in lemma 4.4, we have the estimates:

$$\begin{aligned} \|D_A(\phi)\|_{L^2(K)_0}^2 + \|b\|_{L^2(X)} \|\phi\|_{L^4(K)_0} &\geq -\delta \|\phi\|_{L^2(K)_0}^2 + \frac{1}{4} \|\phi\|_{L^4(K)_0}^4 \\ &\geq -C_K^2 \delta \|\phi\|_{L^4(K)_0}^2 + \frac{1}{4} \|\phi\|_{L^4(K)_0}^4 \end{aligned}$$

In particular there is  $c = c(\|\varphi\|_{L^2}, \|b\|_{L^2}, \delta)$  so that the bound  $\|\phi\|_{L^4(K)_0} \leq c$  holds, and hence  $\|\phi\|_{L^2(K)_0} \leq c'$ .

The rest of the argument is the same as step 2 in lemma 4.4 for this case.

**Step 2:** Let us consider the general case, and choose a solution  $[A_0, \psi_0]$  to the Seiberg-Witten equation over  $M$ . It follows from proposition 4.9 that both  $\|a\|_{L^\infty}$  and  $\|\phi\|_{L^\infty}$  are bounded by a constant which depend on  $r$ . Let us consider the equality  $d^+(a) = b + \sigma(\tilde{\psi}_0, \phi)$ . Recall that support of  $\phi \subset K$ , and hence the equality  $d^+(a) = b$  holds on  $K^c$ . Then we have the estimates:

$$\begin{aligned} \|d^+(a)\|_{L^2(X)}^2 &= \|d^+(a)\|_{L^2(K)}^2 + \|d^+(a)\|_{L^2(K^c)}^2 \\ &= \|b + \sigma(\tilde{\psi}_0, \phi)\|_{L^2(K)}^2 + \|b\|_{L^2(K^c)}^2 \\ &\leq C \|b + \sigma(\tilde{\psi}_0, \phi)\|_{L^\infty(K)}^2 + \|b\|_{L^2(K^c)}^2 \leq C'(r) \end{aligned}$$

where we used  $L^\infty$  bound of  $\phi$ . So we obtain  $L_1^2$  bound:

$$\|a\|_{L_1^2(X)} \leq c(\|d^+(a)\|_{L^2(X)} + \|a\|_{\text{harm}}) \leq c(r).$$

For  $\phi$ , we have the estimates:

$$\begin{aligned} \|D_{\tilde{A}_0}(\phi)\|_{L^2}^2 &\leq 4\|D_A(\phi)\|_{L^2}^2 + 4\|a\phi\|_{L^2}^2 \\ &= 4\|\varphi - a\tilde{\psi}_0\|_{L^2}^2 + 4\|\phi\|_{L^\infty} \|a\|_{L^2}^2 \\ &\leq 4\|\varphi\|_{L^2}^2 + C\|a\|_{L^2}^2 + C\|\varphi\|_{L^2}^2 \|a\|_{L^2}^2 + 4\|\phi\|_{L^\infty} \|a\|_{L^2}^2 \leq c(r) \end{aligned}$$

Since  $\|\phi\|_{L^2} \leq C\|\phi\|_{L^\infty} \leq c(r)$  holds, we obtain  $L_1^2$  bound of  $\phi$ .  $\square$

**4.5. Effect on smallness of local norms on one forms.** It is of interest for us to see how local Sobolev norms effect to its global norm. In particular it is characteristic of non compact space that both situations can happen where local norms are small but total norm is quite large. Below we induce bounds on Sobolev norms under smallness of local norms on one forms.

Let  $K \subset X$  be a fundamental domain. Let us take an element  $(\phi, a)$  and put  $\tilde{\mu}(\phi, a) = (\varphi, b, h)$ .



**Lemma 4.11.** *Let us choose a reducible base solution  $(A_0, 0)$  over  $M$ . Suppose that the AHS complex has closed range, and the Dirac operator is invertible.*

*Then there is small  $\epsilon_0 > 0$  such that  $L^2_{\frac{k+1}{2}}$  bound:*

$$(\phi, a) \in L^2_{\frac{k+1}{2}}(X)$$

*holds in terms of  $\|(\varphi, b, h)\|_{L^2_{\frac{k-1}{2}}}$  and  $K'$ , if the local bounds:*

$$\|a\|_{L^2_{k-1}(\gamma(K))} < \epsilon_0$$

*hold for any  $\gamma \in \Gamma$ .*

*Proof. Step 1:* Let us denote  $A = \tilde{A}_0 + a$ . We claim that there is a positive  $\epsilon_0 > 0$  such that if  $(a, \phi) \in \tilde{\mu}^{-1}(D_r) \subset L^2_k((X, g); \tilde{S}^+ \oplus \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^*$  satisfies the estimates:

$$\|(a, \phi)\|_{L^2_{k-1}(\gamma(K))} < \epsilon_0$$

for all  $\gamma \in \Gamma$ , then  $(a, \phi)$  must satisfy a uniform bound in  $L^2_k(X)$ . In particular  $L^\infty$  bound holds.

Let us check that the estimate:

$$\|a\phi\|_{L^2_{k-1}(X)} \leq C\epsilon_0 \max(\|a\|_{L^2_{k-1}(X)}, \|\phi\|_{L^2_{k-1}(X)})$$

holds. Let  $r_a = \|a\|_{L^2_{k-1}(X)}$ ,  $r_\phi = \|\phi\|_{L^2_{k-1}(X)}$ ,  $r_{\min} = \min(r_a, r_\phi)$  and  $r_{\max} = \max(r_a, r_\phi)$ . It follows from lemma 3.14 that the estimate holds:

$$\begin{aligned} \|a\phi\|_{L^2_{k-1}(X)} &= r_a r_\phi \left\| \frac{a}{r_a} \frac{\phi}{r_\phi} \right\|_{L^2_{k-1}(X)} \leq r_a r_\phi C r_{\min}^{-1} \epsilon_0 \\ &= C\epsilon_0 r_{\max} = C\epsilon_0 \max(r_a, r_\phi). \end{aligned}$$

Let us put  $r = \|(\varphi, b, h)\|_{L^2_{k-1}}$ , and divide into two cases. Suppose  $r_{\max} = r_\phi$ . Notice  $D_{\tilde{A}_0}(\phi) + a\phi$  has  $L^2_{k-1}$  norm less than  $r$ . Then we obtain the estimates:

$$\begin{aligned} \|\phi\|_{L^2_k(X)} &\leq C' \|D_{\tilde{A}_0}(\phi)\|_{L^2_{k-1}(X)} \leq C' (\|D_A(\phi)\|_{L^2_{k-1}(X)} + \|a\phi\|_{L^2_{k-1}(X)}) \\ &\leq C'(r + C\epsilon_0 \|\phi\|_{L^2_{k-1}(X)}). \end{aligned}$$

In particular if  $\epsilon_0 > 0$  is sufficiently small, then we have the estimate:

$$\|\phi\|_{L^2_k(X)} \leq C'' r.$$

Next it follows from lemma 3.8 that the estimates:

$$\|\sigma(\phi)\|_{L^2_{k-1}} \leq \|\phi\|_{L^4_{k-1}} \leq C \|\phi\|_{L^2_k}$$

hold. So we have the estimates:

$$\|d^+(a)\|_{L^2_{k-1}} \leq r + \|\sigma(\phi)\|_{L^2_{k-1}} \leq r + C \|\phi\|_{L^2_k} \leq Cr$$

since  $d^+(a) + \sigma(\phi)$  has  $L_{k-1}^2$  norm less than  $r$ . It follows from the assumption that the estimate holds:

$$\|a\|_{L_k^2} \leq C(\|d^+(a)\|_{L_{k-1}^2} + \|a_{harm}\|) \leq Cr.$$

**Step 2:** Next suppose  $r_{\max} = r_a$ . It follows from lemma 3.14 that the estimate holds:

$$\|\sigma(\phi)\|_{L_{k-1}^2} \leq C\epsilon_0\|\phi\|_{L_{k-1}^2}$$

By a similar argument as step 1, we obtain the estimates:

$$\|d^+(a)\|_{L_{k-1}^2} \leq r + \|\sigma(\phi)\|_{L_{k-1}^2} \leq r + C\epsilon_0\|\phi\|_{L_{k-1}^2} \leq r + C\epsilon_0\|a\|_{L_{k-1}^2}.$$

So we have the estimates:

$$\begin{aligned} \|a\|_{L_k^2} &\leq C(\|d^+(a)\|_{L_{k-1}^2} + \|a_{harm}\|) \\ &\leq C(r + C\epsilon_0\|a\|_{L_{k-1}^2} + \|a_{harm}\|) \leq C'(r + C\epsilon_0\|a\|_{L_k^2}). \end{aligned}$$

So if  $\epsilon_0 > 0$  is sufficiently small, then we have the bounds:

$$\|a\|_{L_k^2} \leq Cr.$$

Since  $r_{\max} = r_a$ , the bound  $\|\phi\|_{L_{k-1}^2} \leq Cr$  also holds. Then we have the estimates:

$$\begin{aligned} \|\phi\|_{L_k^2} &\leq c\|D_{\tilde{A}_0}(\phi)\|_{L_{k-1}^2} \leq c(\|b\|_{L_{k-1}^2} + \|a\phi\|_{L_{k-1}^2}) \\ &\leq c\|b\|_{L_{k-1}^2} + C\epsilon_0\|a\|_{L_{k-1}^2} \leq Cr. \end{aligned}$$

This verifies the claim in step 1.

**Step 3:** Now suppose  $\|(a, \phi)\|_{L_{k-1}^2(\gamma(K))} \geq \epsilon_0$  hold for some  $\gamma \in \Gamma$  (and hence  $\|\phi\|_{L_{k-1}^2(\gamma(K))} \geq \epsilon_0$  should hold by the assumption). Let us induce a weak version to step 1. Let us put  $l = \frac{k-1}{2}$ . It follows from corollary 3.15 that the estimate holds:

$$\|a\phi\|_{L_l^2(X)} \leq C\epsilon_0\|\phi\|_{L_l^2(X)}.$$

Then we obtain a weak version of the uniform bounds:

$$\|a\|_{L_l^2(X)}, \|\phi\|_{L_l^2(X)} \leq C\|(\varphi, b, h)\|_{L_{\frac{k-1}{2}}^2(X)}$$

by the same argument as step 1. □

## 5. APPROXIMATION BY FINITE DIMENSIONAL SPACES

**5.1. Fredholm map.** Let  $H'$  and  $H$  be two Hilbert spaces, and consider a Fredholm map:

$$F = l + c : H' \rightarrow H$$

whose linear part  $l$  is Fredholm and  $c$  is compact. For our purpose later, we restrict the domain space  $H' = L_k^2(X; E)$  by the Sobolev space over some vector bundle  $E \rightarrow X$ . Analysis of finite dimensional approximation of  $F$  has been developed for a metrically proper and Fredholm map [S], [BF]. In particular it is applied to the monopole map when  $X$  is compact.

Below we shall introduce an equivariant version of such non linear map over covering spaces  $X = \tilde{M}$  of compact four manifolds.

Let us say that a smooth map  $c : H' = L_k^2(X; E) \rightarrow H$  is *locally compact*, if its restriction  $c|L_k^2(K, E)_0$  over any compact subset is compact.

**5.2. Technical estimates.** The results in this subsection are applied to perform finite dimensional approximations in the next subsection. In particular the lemma below is applied to the Dirac operator in the case of the covering monopole map.

Here is a technical lemma. Let:

$$D : L_k^2(X; E) \rightarrow L_{k-1}^2(X; F)$$

be a first order elliptic differential operator. Let  $K \subset X$  be a compact subset, and consider the restriction:

$$D : L_k^2(K; E)_0 \rightarrow L_{k-1}^2(K; F)_0$$

**Lemma 5.1.** *Suppose  $D : L_k^2(X; E) \rightarrow L_{k-1}^2(X; F)$  is surjective with closed range. Then:*

(1) *it is equivalent to that  $D^* : L_k^2(X; F) \rightarrow L_{k-1}^2(X; E)$  is injective with closed range.*

(2)  *$D : L_k^2(K; E)_0 \rightarrow L_{k-1}^2(K; F)_0$  is also surjective with closed range for any  $K \subset X$ .*

*Proof. Step 1:* Let us check that the formal adjoint

$$D^* : L_k^2(X; F) \rightarrow L_{k-1}^2(X; E)$$

is injective with closed range.

If  $w \in L_k^2(X; F)$  with  $D^*(w) = 0$  could hold, then

$$\langle D^*(w), u \rangle_{L_{k-1}^2} = \langle w, D(u) \rangle_{L_{k-1}^2} = 0$$

follow for all  $u \in L_k^2(X; E)$ . This implies  $w = 0$  by surjectivity of  $D$ .

There is a constant  $C$  with the bound  $\|D(u)\|_{L_{k-1}^2} \geq C\|u\|_{L_k^2}$ . For any  $w \in L_k^2$ , there is  $u' \in L_{k+1}^2$  such that  $w = D(u')$  holds. Then

$$\begin{aligned} \|D^*(w)\|_{L_{k-1}^2}^2 &= \langle D^*(w), D^*(w) \rangle_{L_{k-1}^2} \\ &= \langle D^*D(u'), D^*D(u') \rangle_{L_{k-1}^2} \geq C\|u'\|_{L_{k+1}^2}^2 \\ &\geq C\|D(u')\|_{L_k^2}^2 = C\|w\|_{L_k^2}^2. \end{aligned}$$

Hence  $D^*$  also has closed range.

**Step 2:** Conversely suppose  $D : L_k^2(X; E) \rightarrow L_{k-1}^2(X; F)$  has closed range but is not surjective. Then there is  $0 \neq u \in L_{k-1}^2(X; F)$  with:

$$\langle D(w), u \rangle_{L_{k-1}^2} = \langle w, D^*(u) \rangle_{L_{k-1}^2} = 0$$

hold for any  $w \in L_k^2(X; E)$ . This implies  $u \in L_k^2(X; F)$  with  $D^*(u) = 0$ , since  $D^*$  is elliptic.

Combining with step 1, it follows under closeness of  $D$  that  $D$  is surjective if and only if  $D^*$  is injective.

**Step 3:** Suppose  $D$  is surjective with closed range, and consider the restriction  $D : L_k^2(K; E)_0 \rightarrow L_{k-1}^2(K; F)_0$ .

$D^* : L_k^2(K; F)_0 \rightarrow L_{k-1}^2(K; E)_0$  is injective with closed range, since  $L_{k-1}^2(K; F)_0 \rightarrow L_k^2(X; F)$  is a closed linear subspace.

Suppose  $D : L_k^2(K; E)_0 \rightarrow L_{k-1}^2(K; F)_0$  is not surjective, and choose  $0 \neq u \in L_{k-1}^2(K; F)_0$  with

$$\langle D(w), u \rangle_{L_{k-1}^2} = \langle w, D^*(u) \rangle_{L_{k-1}^2} = 0$$

for any  $w \in L_k^2(K; E)_0$ .

Let  $u_i \in C_c^\infty(K; F)$  which converge to  $u$  in  $L_{k-1}^2$ . Then

$$\lim_{i \rightarrow \infty} \langle w, D^*(u_i) \rangle_{L_{k-1}^2} = 0$$

hold for any  $w \in L_k^2(K; E)_0$ . But actually the convergence above holds for any  $w \in L_k^2(X; E)$ , since  $D^*(u_i) \in C_c^\infty(K; F)$ . So  $u_i$  converge to  $u$  in  $L_{k-1}^2$  and  $D^*(u_i)$  weakly converge to 0 in  $L_{k-1}^2$ . Thus  $u \in L_{k-1}^2(K; F)_0 \cap L_k^2(X; F)$  with  $D^*(u) = 0$  by elliptic regularity. This contradicts to step 1. □

**Lemma 5.2.** *Suppose a sequence  $w_i \in L_k^2(X; F)$  with  $\|w_i\|_{L_k^2} = 1$  satisfies a condition:*

$$\lim_{i \rightarrow \infty} \sup_{v \in B \cap \text{im } D} |\langle w_i, v \rangle_{L_{k-1}^2}| = 0$$

where  $B \subset L_{k-1}^2(X; F)$  is the unit ball.

Then  $D^*(w_i)$  or  $w_i$  converge to 0 in  $L_{k-1}^2$ .

In particular  $w_i \rightarrow 0$  holds in  $L_{k-1}^2$ , if  $D$  is surjective with closed range.

*Proof.* Assume that there is a sequence  $w_i \in L_k^2(X; F)$  with  $\|w_i\|_{L_{k-1}^2} = 1$  and:

$$\sup_{v \in B \cap \text{im } D} | \langle w_i, v \rangle_{L_{k-1}^2} | < \epsilon_i \rightarrow 0$$

Then any  $f \in L_k^2(X; E)$  with  $\|D(f)\|_{L_{k-1}^2} = 1$  satisfy the bounds:

$$\langle D^*(w_i), f \rangle_{L_{k-1}^2} = \langle w_i, D(f) \rangle_{L_{k-1}^2} < \epsilon_i$$

by lemma 3.1.

Suppose there is a lower bound  $\|D^*(w_i)\|_{L_{k-1}^2} \geq c > 0$ . Let  $P$  be the spectral projection of  $DD^*$  on  $L^2(X; F)$ . Then:

$$\|P_{[0, \lambda^2)}(w_i)\|_{L_{k-1}^2} < \epsilon_i \rightarrow 0$$

must hold for any  $\lambda > 0$ , otherwise such situation cannot happen.

So we obtain the estimates:

$$\begin{aligned} \|w_i\|_{L_{k-1}^2} &\leq \|(1 - P_{[0, \lambda^2)})(w_i)\|_{L_{k-1}^2} + \|P_{[0, \lambda^2)}(w_i)\|_{L_{k-1}^2} \\ &\leq \lambda^{-1} \|w_i\|_{L_k^2} + \epsilon_i = \lambda^{-1} + \epsilon_i \rightarrow 0 \end{aligned}$$

$L_{k-1}^2$  norms of the sequence converges to 0 as  $\lambda \rightarrow \infty$ .

The last statement follows by lemma 5.1(1). □

**5.3. Finite dimensional approximations.** To apply method of finite dimensional approximation, we need to induce a kind of properness on the image of the projection.

Let  $F = l + c : H' \rightarrow H$  be a metrically proper map between Hilbert spaces so that  $\|F(m)\| \geq \delta \|m\|$  hold for some  $\delta > 0$ .

**Lemma 5.3.** *Suppose  $l$  is surjective and  $c$  is compact. Then for any  $r > 0$ , there is a finite dimensional linear subspace  $W' \subset H'$  such that*

$$pr \circ F : D_r \cap W' \rightarrow W$$

*also satisfies the bound:*

$$\|pr \circ F(m)\| \geq \frac{\delta}{2} \|m\|$$

*for any  $m \in D_r$ .*

*Proof.* Let  $C \subset H$  be the closure of the image  $c(D_r)$ , which is compact. Hence there is a finite number of points  $w_1, \dots, w_m \in c(D_r)$  such that its  $\frac{\delta}{2}$  neighborhood covers  $C$ .

Choose  $w'_i \in H'$  so that  $l(w'_i) = w_i$  hold for  $1 \leq i \leq m$ , and let  $W'$  as the linear span of these  $w'_i$ . The restriction  $\text{pr} \circ F : D_r \cap W' \rightarrow W$  is the desired one.  $\square$

**Corollary 5.4.** *Let:*

$$F = l + c : L_k^2(X : E) \rightarrow L_{k-1}^2(X : F)$$

*be a smooth map where  $l$  is surjective with closed range, and  $c$  is locally compact.*

*Let  $K \subset X$  be a compact subset, and consider the restriction:*

$$F : L_k^2(K : E)_0 \rightarrow L_{k-1}^2(K : F)_0.$$

*Then for any  $r > 0$ , there is a finite dimensional linear subspace  $W' \subset L_k^2(K : E)_0$  such that*

$$\text{pr} \circ F : D_r \cap W' \rightarrow W = l(W)$$

*also satisfies the bound:*

$$\|\text{pr} \circ F(m)\| \geq \frac{\delta}{2} \|m\|$$

*for any  $m \in D_r$ .*

*Proof.* This follows from lemma 5.3 with lemma 5.1(2).  $\square$

Notice that if  $l$  is not injective, then  $l^{-1}(W')$  is already infinite dimensional by infinite group symmetry.

Let us describe what we can induce in the case of injective  $l$  by use of the results in the previous subsection.

**Lemma 5.5.** *Let:*

$$F = l + c : L_k^2(X; E) \rightarrow L_{k-1}^2(X; F)$$

*be a smooth map where  $l$  is a first order elliptic differential operator and  $c$  is quadratic which is locally compact.*

*If  $l$  is injective with closed range, then the composition of the projection with  $F$ :*

$$\text{pr} \circ F : W' \rightarrow W$$

*is proper on any finite dimensional linear subspace  $W' \subset L_k^2(X; E)$ , where  $\text{pr} : L_{k-1}^2(X; E) \rightarrow W = l(W')$ .*

*Proof.* There is a constant  $C > 0$  such that  $\|l(u)\|_{L_{k-1}^2} \geq C\|u\|_{L_k^2}$  holds for  $u \in L_k^2(X; E)$ . Moreover  $\|c(u)\|_{L_k^2} \leq C\|u\|_{L_k^2}$  also holds.

Let us take a unit element  $u \in W' \subset L_k^2(X; E)$  with  $\|u\|_{L_{k-1}^2} = 1$ . If  $c(u) = 0$ , then there is nothing to do more, and so suppose  $\|c(u)\|_{L_k^2} \neq 0$ . Let us divide into two cases;

(1) Suppose  $\sup_{v \in B \cap \text{im } l} | \langle c(u), v \rangle_{L_{k-1}^2} | < \epsilon$  is sufficiently small. Then at least one of  $\|D^*(c(u))\|_{L_{k-1}^2} \|c(u)\|_{L_{k-1}^2}$  is small by lemma 5.2.

In the former case, it contains non zero harmonic component  $\text{Ker } D^*$ , and hence  $c(u)$  is linearly independent of  $l(W)$ .

Suppose the latter case. Its spectral distribution leans toward high level. So  $c(u)$  and any vector in  $l(W)$  are also linear independent.

In both cases,  $\|F(tu)\|_{L_{k-1}^2} \rightarrow \infty$  as  $t \rightarrow \infty$  since  $D$  is injective.

(2) Suppose  $\sup_{v \in B \cap \text{im } D} | \langle c(u), v \rangle_{L_{k-1}^2} | > \epsilon_0$  is positive. Then it follows from the equality:

$$F(tu) = tD(u) + t^2c(u)$$

that  $\|F(tu)\|_{L_{k-1}^2} \rightarrow \infty$  also holds as  $t \rightarrow \infty$ . □

**Remark 5.6.** *In the case of covering monopole maps we analyze, the domain is not the full Sobolev space, rather its closed linear subspace  $L_k^2(K; \tilde{S}^+)_0 \oplus L_k^2(X; \Lambda^1) \cap \text{Ker } d^*$ . Moreover the target space is sum of the Sobolev space with the first  $L^2$  cohomology group. Still the contents in section 5 work for this case. In particular corollary 5.4, lemma 5.5, corollary 5.7 and proposition 5.10 below also hold for the covering monopole maps, since the linearized map splits into the sum of the Dirac operator with  $d^+$  and the harmonic projection.*

Let  $s_0 < s_1 < \dots \rightarrow \infty$  be an increasing sequence, and consider a family of maps:

$$F_i : D_{s_i}(W'_i) \rightarrow H$$

where  $D_{s_i} \subset W'_i \subset H'$  are balls with radii  $s_i$  in finite dimensional linear subspaces  $W'_i$  such that the union:

$$\bigcup_{i=0}^{\infty} W'_i \subset H'$$

is dense.

Let us say that the family  $\{F_i\}_i$  is *asymptotically proper* on  $H'$ , if there is another sequence  $r_0 < r_1 < \dots \rightarrow \infty$  such that the embeddings hold:

$$F_i^{-1}(D_{r_i}) \subset D_{s_i}(W_i).$$

By use of the terminology, one can state a combination of corollary 5.4 with lemma 5.5 as below:

**Corollary 5.7.** *Let:*

$$F = l + c : L_k^2(X; E) \rightarrow L_{k-1}^2(X; F)$$

*be a locally strongly proper map where  $l$  is a first order elliptic differential operator and  $c$  is quadratic which is locally compact.*

Suppose  $l$  has closed range, and satisfies at least one of injectivity or surjectivity. Let us consider the restriction:

$$F = l + c : L_k^2(K; E)_0 \rightarrow L_{k-1}^2(K; F)_0$$

over a compact subset  $K \subset X$ .

Then there exists a family of finite dimensional linear subspaces  $W'_i \subset L_k^2(K; E)_0$  and an asymptotically proper maps on  $L_k^2(K; E)_0$ :

$$F_i \equiv \text{pr}_{W_i} \circ F : D_{s_i}(W'_i) \rightarrow L_{k-1}^2(K; F)_0$$

where  $\text{pr}_{W_i} : H \rightarrow W_i \equiv l(W'_i) \subset L_{k-1}^2(K; F)_0$  are the orthogonal projections.

**Remark 5.8.** There are two opposite properties on possible choices for these data above. If  $l$  is injective, then  $s_i$  depend on  $W_i$ , while the converse holds if  $l$  is surjective.

In particular one may replace  $s_i$  by larger number for the former case, while may replace  $W_i$  by another  $U_i \supset W_i$  for the latter.

**5.4. Compactly supported families of Sobolev spaces.** Let  $K_1 \subset \subset \dots \subset \subset K_i \subset \subset K_{i+1} \subset X$  be an exhaustion of  $X$  by compact subsets. Let  $F_i = \text{pr}_{W_i} \circ F$  be as above.

**Lemma 5.9.** For any  $v' \in L_k^2(X; E)$ , there are finite dimensional linear subspaces  $W'_i \subset L_{k-1}^2(K_i; E)_0$  such that any approximations  $v'_i \in W'_i$  with  $v'_i \rightarrow v'$  in  $L_k^2(X; E)$  satisfy Cauchy:

$$\lim_{i,j} \|F_i(v'_i) - F_j(v'_j)\|_{L_{k-1}^2} \rightarrow 0$$

*Proof.* Let  $v'_i \in L_k^2(K_i; E)_0$  be any approximation with  $v'_i \rightarrow v'$ , and put  $w_i \equiv c(v'_i) \in L_{k-1}^2(K_i)_0$  where  $P_i : L_{k-1}^2(K_i)_0 \rightarrow W_i$  are the orthogonal projections. Choose  $W_i$  so that they include both  $w_i$  and  $l(v'_i)$ . Then the family of subspaces  $W_i$  satisfies the conclusion for this particular family  $\{v'_i\}_i$ .

On the other hand another other approximation  $u'_i$  of  $v'$  also satisfies the same property, since their differences converge to zero  $\|v'_i - u'_i\| \rightarrow 0$  as  $i \rightarrow \infty$ .

□

Notice that  $\{W_i\}_i$  above in the proof depends on choice of  $v'$ .

**Definition 5.1.** Let  $F : H' \rightarrow H$  be a smooth map.



Let us say that an increasing family of finite dimensional linear subspaces and their open subsets:

$$\begin{array}{ccccccc} W'_0 & \subset & W'_1 & \subset & \dots & \subset & W'_i & \subset & \dots \\ \cup & & \cup & & & & \cup & & \\ D'_0 & \subset & D'_1 & \subset & \dots & \subset & D'_i & \subset & \dots \end{array}$$

weakly approximate  $F$ , if

- (1) their union  $\cup_{i \geq 0} D'_i \subset H'$  is dense, and
- (2) for any  $v' \in H'$  and any approximation  $v'_i \in D'_i$  with  $\|v'_i - v'\| \rightarrow 0$ , the above property holds:

$$\lim_{i,j} \|F_i(v'_i) - F_j(v'_j)\|_H \rightarrow 0$$

where:

$$F_i \equiv \text{pr}_i \circ F : D'_i \rightarrow W_i$$

and  $\text{pr}_i : H \rightarrow W_i$  are the orthogonal projections.

- (3) The above data with  $\{F_i\}_i$  consists of an asymptotically proper family.

Suppose both  $H'$  and  $H$  admits isometric actions by a group  $\Gamma$ , and assume that  $F$  is  $\Gamma$  equivariant. Then we say that the above families weakly  $\Gamma$ -approximate  $F$ , if moreover they satisfy:

$$\lim_{i,j} \|F_i(\gamma(v'_i)) - \gamma(F_i(v'_i))\|_H \rightarrow 0$$

for any  $\gamma \in \Gamma$ .

**Proposition 5.10.** *Let:*

$$F = l + c : H' = L_k^2(X; E) \rightarrow H = L_{k-1}^2(X; F)$$

be a  $\Gamma$ -equivariant locally strongly proper map, where  $l$  is a first order elliptic differential operator and  $c$  is quadratic which is locally compact. Suppose  $l$  has closed range, and satisfies at least one of injectivity or subjectivity.

Then there is a family of finite dimensional linear subspaces  $\{W'_i\}_i$  which weakly  $\Gamma$ -approximates  $F$ .

*Proof. Step 1:* Let  $K_1 \subset \dots \subset K_i \subset K_{i+1} \subset X$  be exhaustion of  $X$  by compact subsets, and fix a decreasing positive sequence  $\epsilon_m > \epsilon_{m+1} > \dots \rightarrow 0$ .

For each  $i$  and  $r > 1$ , there is  $s = s_{r,i}$  and finite dimensional linear subspaces  $W'_{i,r} \subset L_k^2(K_i; E)_0$  such that the inclusions:

$$(\text{pr}_{i,r} \circ F)^{-1}(D_r \cap W_{i,r}) \subset D_s \cap W'_{i,r}$$

hold where  $\text{pr}_{i,r} : H \rightarrow W_{r,i}$  are the orthogonal projections, and  $\lim_{r \rightarrow \infty} s_{r,i} = \infty$  holds for each  $i$ . Moreover the union  $\cup_r W'_{r,i} \subset L_k^2(K_i; E)$  is dense.

**Step 2:** Let us choose  $r_m$  with  $\lim_m r_m = \infty$ , and consider  $W_{r_m,i} \subset L_k^2(K_i; E)_0$ . Let  $\text{pr}_{m,i} : H \rightarrow W_{r_m,i}$  be the orthogonal projections, and put  $F_{m,i} \equiv \text{pr}_{m,i} \circ F$ . Then the inclusions:

$$F_{m,i}^{-1}(D_{r_m} \cap W_{i,r_m}) \subset D_{s_m} \cap W'_{i,r_m}$$

hold, where  $\lim_{m \rightarrow \infty} s_m = \infty$ . Notice that:

$$\lim_{m,m' \rightarrow \infty} \|F_{m,i}(v') - F_{m',i}(v')\|_H = 0$$

hold for any  $v' \in L_k^2(K_i; E)_0$ . Actually the equalities:

$$\lim_{m,m' \rightarrow \infty} F_{m,i}(v') = (l + \text{pr}_i \circ c)(v')$$

hold where  $\text{pr}_i : H \rightarrow l(L_k^2(K_i; E)) \subset L_{k-1}^2(K_i; E)$  are the orthogonal projections.

**Step 3:** Let us fix an increasing sequence  $s_m \rightarrow \infty$ . Suppose we have chosen the family  $r_m = r_m^i$  and  $s_m^i$  over  $K_i$  in step 2, so that

$$(\text{pr}_{i,r_m^i} \circ F)^{-1}(D_{r_m^i} \cap W_{i,r_m^i}) \subset D_{s_m^i} \cap W'_{i,r_m^i}$$

where  $\text{pr}_{i,r_m^i} : H \rightarrow W_{r_m^i,i}$  are the orthogonal projections. We can choose them as above for all  $i$  so that the same things hold as step 1 and 2 over  $K_{i+1}$  by replacing  $r_m^i, s_m^i$  by  $r_m^{i+1}$  and  $s_m^{i+1}$  respectively.

One may choose both constants and families  $\{W'_{m,i}\}_m \subset L_k^2(K_i; E)_0$  inductively on  $i$  so that the following properties hold:

$$\begin{aligned} r_m^{i+1} &\geq r_m^i, & s_m^{i+1} &\geq s_m^i, \\ W'_{r,i} &\subset W'_{m,i+1} \subset L_k^2(K_{i+1}; E)_0. \end{aligned}$$

Then for any  $v' \in L_k^2(K_i; E)_0$ , the convergence holds:

$$\lim_{m,m',j,j' \rightarrow \infty} \|F_{m,j}(v') - F_{m',j'}(v')\|_H \rightarrow 0.$$

Actually the limit exists:

$$\lim_{m,m',j,j' \rightarrow \infty} F_{m,j}(v') = \begin{cases} F(v') & l \text{ is surjective,} \\ (l + \text{pr} \circ c)(v') & l \text{ is injective} \end{cases}$$

where  $\text{pr} : H \rightarrow l(L_k^2(X; E)) \subset L_{k-1}^2(X; F)$  is the orthogonal projection.

Notice that  $\lim_i (F_{m,i} - F_{m,i+1})$  may not converge in the operator norm, since the linear subspaces  $\text{pr}_i(c(L_k^2(K_i; E)_0 \cap D_{r_m^i}))$  may depend on  $i$  for some  $m$  (see 5.5).

Let us restrict the indices  $m = i$  so that  $r_i^i \equiv r_i$  and  $s_i^i = s_i$  and so on, and put:

$$F_i \equiv \text{pr}_{i,i} \circ F : W'_i \cap D_{r_i} \rightarrow W_i \subset H$$

for  $W'_i \equiv W'_{i,r_i^i} \subset L_k^2(K_i; E)_0$ . These data weakly approximate  $F$ .

Notice that:

$$\lim_{i,j} \|F_i(v'_i) - F_j(v'_j)\|_H = 0$$

holds for any  $v' \in L_k^2(X; E)$  and its approximation  $v'_i \in W'_i \cap D_{r_i}$ , since  $\lim_i \|v'\|_{L_k^2(K_i^c)} = 0$  holds.

**Step 4:** Let us consider  $\Gamma$ -equivariance, and  $\text{pr}'_i : H \rightarrow \gamma(W_i)$  be the orthogonal projections. Then for any  $u \in H$ ,

$$\lim_i \|\text{pr}'_i(u) - \text{pr}_i(u)\|_H = 0$$

holds, since their union  $\cup_i W_i \subset \text{im } l$  is dense and  $l$  is  $\Gamma$ -equivariant.

Let us replace all subspaces by  $\gamma$  action so that  $W'_i$  by  $\gamma(W'_i)$  and so on. Then the families  $\{(\gamma(W'_i), \gamma(D'_i))\}_i$  also weakly approximate  $F$ , and their limits coincide:

$$\lim_i \|\text{pr}'_i \circ F(\gamma(v'_i)) - F_i(\gamma(v'_i))\|_H = 0$$

for any  $v' \in L_k^2(X; E)$  and its approximations  $v'_i \in L_k^2(K_i; E)_0$ . On the other hand we have the equalities:

$$\text{pr}'_i \circ F(\gamma(v'_i)) = \text{pr}'_i(\gamma(F(v'_i))) = \gamma(\text{pr}_i \circ F(v'_i)).$$

This verifies  $\Gamma$ -equivariance of weak approximation of  $F$ .

So we have obtained the desired families of finite dimensional linear subspaces and their open subsets by  $\{W_i\}_i$  and  $D'_i \equiv W'_i \cap D_{r_i}$  respectively, which weakly  $\Gamma$ -approximates  $F$ . □

**5.5. Sliding end phenomena.** In the above construction,  $F_i|_{D_{s_i}(W'_i)}$  may not converge to  $F$  in operator topology, when one finds difference between the images of  $\text{pr}(c(L_k^2(K)_0)) \subset L_{k-1}^2(K)_0$  and  $l(L_k^2(K)_0)$ .

Let us recall the proof in lemma 5.3. Let  $K \subset X$  be a compact subset, and consider the image:

$$c(D_r \cap L_k^2(K)_0) \subset L_{k-1}^2(X)$$

which is relatively compact. So choose a finite number of vectors  $w_1, \dots, w_m \in c(D_r \cap L_k^2(K)_0)$  which consists of  $\epsilon$ -net.

Suppose  $w'_i = l^{-1}(w_i)$  exist, and put the finite dimensional linear subspace  $V' \subset L_k^2(X)$  spanned by vectors  $w'_i$  with  $V = l(V')$ . Surely  $w'_i \in L_k^2(X; E)$  but they may not be in  $L_k^2(K)_0$  in general, so  $c(w'_i)$  may

be away from  $c(D_r \cap L_k^2(K)_0)$ . Hence the composition  $\text{pr} \circ F : V' \rightarrow V$  of  $F$  with the projection to  $V$  may not be close to  $F$  strongly even if  $\dim V$  is large.

**Example 5.11.** *Let us give a simple example. Let  $H' = H = l^2(\mathbb{Z})$ , and consider:*

$$F = l + c : H' \rightarrow H$$

where  $l(\{a_i\}_i) = \{a_{i-1}\}_i$  is the shift, and  $c(\{a_i\}) = \{b_i\}_i$  with  $b_i = f(a_i)$ .

Let us put  $V_{m,n} = \{ \{a_i\}_i : a_i = 0 \text{ for } i \leq m \text{ or } n \leq i \}$ . Then  $l : V_{m,n} \cong V_{m+1,n+1}$  and the restriction  $\text{pr} \circ F - F : V_{m,n} \rightarrow V_{m,n+1}$  satiafies:

$$(\text{pr} \circ F - F)(\{a_i\}_i) = -f(a_m).$$

So  $\text{pr} \circ F$  pushes bubbling  $f(a_m)$  off as  $m \rightarrow -\infty$ .

Let us introduce a sliding end quantity. Let  $K_1 \subset \subset \cdots \subset \subset K_i \subset \subset K_{i+1} \subset X$  be an exhaustion of  $X$  by compact subsets, and

$$\text{pr}_i : L_{k-1}^2(K_i)_0 \rightarrow l(L_k^2(K_i)_0)$$

be the orthogonal projections.

Let us introduce a *sliding end quantity*  $b(F) \in [0, C_0]$  which is given by:

$$b(F) \equiv \inf_{\{K_i\}_i} \lim_{i \rightarrow \infty} b(F)_i$$

where  $b(F)_i = \sup_{v \in L_k^2(K_i)_0} \{ \|(1 - \text{pr}_i)(c(v))\|_{L_{k-1}^2} : \|v\|_{L_k^2(K_i)_0} \leq 1 \}$ .

## 6. INFINITE DIMENSIONAL BOTT PERIODICITY

Let  $\Gamma = \pi_1(M)$  be the fundamental group of a compact four manifold  $M$ , and  $H, H'$  be function spaces over the universal covering space  $X = \tilde{M}$ . Let  $F = l + c : H' \rightarrow H$  be a  $\Gamma$  equivariant smooth map whose linear part is  $l$ . What we want to construct is an ‘induced map’ between these function spaces. Of course one can not obtain such a map in the usual sense, since these spaces are locally non compact. This is the reason why we shall introduce the infinite dimensional Clifford  $C^*$ -algebras by Higson-Kasparov-Trout.

**6.1. Quick review of HKT construction.** In 6.1 we quickly review the construction of the Hilbert space Clifford  $C^*$ -algebras [HKT].

Let  $E$  be a finite dimensional Euclidean space, and let  $Cl(E)$  be the complex Clifford algebras, where we choose positive sign on the multiplication  $e^2 = |e|^2 1$  for every  $e \in E$ .

It admits a natural  $\mathbb{Z}_2$ -grading. The embedding  $C : E \rightarrow Cl(E)$  gives a map which is called the Clifford operator. Let us denote  $\mathfrak{C}(E) =$

$C_0(E, Cl(E))$ . Let  $X : C_0(\mathbb{R}) \rightarrow C_0(\mathbb{R})$  be given by  $X(f)(x) = xf(x)$ .  $C_0(\mathbb{R})$  also admits a natural  $\mathbb{Z}_2$ -grading by even or odd functions.

Both operators  $C$  and  $X$  are degree one and essentially self adjoint unbounded multipliers on  $\mathfrak{C}(E)$  and  $C_0(\mathbb{R})$  respectively. In particular  $X \hat{\otimes} 1 + 1 \hat{\otimes} C$  is a degree one and essentially self adjoint unbounded multiplier on  $C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}(E)$ .

Let us introduce a  $*$ -homomorphism:

$$\beta : C_0(\mathbb{R}) \rightarrow S\mathfrak{C}(E) \equiv C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}(E)$$

defined by:

$$\beta : f \rightarrow f(X \hat{\otimes} 1 + 1 \hat{\otimes} C)$$

through functional calculus.

Let  $E$  be a separable real Hilbert space,  $E_a \subset E_b \subset E$  be a pair of finite dimensional linear subspaces, and denote the orthogonal complement as  $E_{ba} = E_b \cap E_a^\perp$ . Then one has the canonical isomorphism  $S\mathfrak{C}(E_b) \cong S\mathfrak{C}(E_{ba}) \hat{\otimes} \mathfrak{C}(E_a)$  of  $C^*$  algebras. Let us introduce a  $*$ -homomorphism passing through this isomorphism:

$$\beta_{ba} = \beta \hat{\otimes} 1 : S\mathfrak{C}(E_a) \rightarrow S\mathfrak{C}(E_{ba}) \hat{\otimes} \mathfrak{C}(E_a) = S\mathfrak{C}(E_b)$$

**Lemma 6.1.** *Let  $E_a \subset E_b \subset E_c$ . Then the composition:*

$$S\mathfrak{C}(E_a) \xrightarrow{\beta_{ba}} S\mathfrak{C}(E_b) \xrightarrow{\beta_{cb}} S\mathfrak{C}(E_c)$$

*coincides with the  $*$ -homomorphism:*

$$\beta_{ca} : S\mathfrak{C}(E_a) \rightarrow S\mathfrak{C}(E_c).$$

**Definition 6.1.** *We denote the direct limit  $C^*$  algebras by:*

$$S\mathfrak{C}(E) = \lim_a S\mathfrak{C}(E_a).$$

It follows from the above construction that one obtains a  $*$  homomorphism:

$$\beta : C_0(\mathbb{R}) \rightarrow S\mathfrak{C}(E).$$

Suppose a discrete group  $\Gamma$  acts on  $E$  by linear and isometrically. Then the action is induced on  $S\mathfrak{C}(E)$ . For a  $\Gamma$   $C^*$ -algebra  $A$ , let us denote:

$$K^\Gamma(A) \equiv K(A \rtimes \Gamma)$$

where the right hand side  $C^*$  algebra is given by the full crossed product of  $A$  with  $\Gamma$ .

**Proposition 6.2 (HKT).** *The  $\Gamma$  equivariant  $*$  homomorphism  $\beta_{ba}$  induces an isomorphism*

$$\beta_* : K^\Gamma(C_0(\mathbb{R})) \cong K^\Gamma(C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}(E)).$$

**6.2. Direct limit  $C^*$  algebras.** Let  $H$  be a Hilbert space on which  $\Gamma$  acts linear isometrically. Choose exhaustion by finite dimensional linear subspaces  $V_j \subset V_{j+1}$  with dense union  $\cup_j V_j \subset H$ . Let  $0 < r_1 < \dots < r_i < r_{i+1} < \dots \rightarrow \infty$  be a divergent positive sequence, and  $D_{r_i}^j \subset V_j$  be the discs with diameter  $r_i$ .

Consider the diagram of the embeddings:

$$\begin{array}{ccccccc}
 & \vdots & & \vdots & & \vdots & \\
 & \cap & & \cap & & \cap & \\
 \subset & D_{r_i}^j & \subset & D_{r_{i+1}}^j & \subset & \dots & \subset V_j \\
 & \cap & & \cap & & & \cap \\
 \subset & D_{r_i}^{j+1} & \subset & D_{r_{i+1}}^{j+1} & \subset & \dots & \subset V_{j+1} \\
 & \cap & & \cap & & & \cap \\
 & \vdots & & \vdots & & & \vdots \\
 \subset & D_{r_i} & \subset & D_{r_{i+1}} & \subset & & \subset H
 \end{array}$$

Let us put:

$$S\mathfrak{C}(D_{r_i}^j) \equiv C_0(\mathbb{R}) \hat{\otimes} C_0(D_{r_i}^j : Cl(V_j))$$

Let  $V_j^\perp \subset H$  be the orthonormal complement, and denote:

$$V_{j,j'} \equiv V_j^\perp \cap V_{j'}, \quad E_{r_i}^{j,j'} = D_{r_i}^j \times V_{j,j'}, \quad E_{r_i}^j = D_{r_i}^j \times V_j^\perp$$

for  $j' \geq j$ . Recall the Bott map:

$$\beta : C_0(\mathbb{R}) \rightarrow S\mathfrak{C}(V) \quad f \rightarrow f(X \hat{\otimes} 1 + 1 \hat{\otimes} C)$$

for a finite dimensional vector space  $V$ . Then we have  $*$  homomorphisms:

$$\begin{aligned}
 \beta_{j,j'} &= \beta \hat{\otimes} 1 : S\mathfrak{C}(D_{r_i}^j) \rightarrow S\mathfrak{C}(E_{r_i}^{j,j'}) \\
 &\cong S\mathfrak{C}(D_{r_i}^j) \hat{\otimes} C_0(V_{j,j'} : Cl(V_{j,j'})) \\
 &\hookrightarrow S\mathfrak{C}(V_{j'}).
 \end{aligned}$$

**Lemma 6.3.** *The direct limit  $C^*$  algebras coincides with the Hilbert space Clifford  $C^*$  algebras:*

$$\lim_{j \geq i} S\mathfrak{C}(D_{r_i}^j) = S\mathfrak{C}(H).$$

*Proof.* Let  $\beta : S\mathfrak{C}(V) \rightarrow S\mathfrak{C}(H)$  be the  $*$  homomorphism into the Clifford  $C^*$  algebra. An element  $a \in S\mathfrak{C}(H)$  is given as  $\lim_j \beta(a_j)$  for some  $a_j \in S\mathfrak{C}(V_j)$ . Let  $\varphi_j \in C_c^\infty(D_{r_j}^j; [0, 1])$  be cut off functions with  $\varphi_j|_{D_{r_{j-1}}^j} \equiv 1$ . We claim that  $b_j \equiv \varphi_j a_j \in S\mathfrak{C}(D_{r_j}^j)$  converges to the same element:

$$\lim_j \beta(b_j) = a \in S\mathfrak{C}(H).$$

Suppose contrary. Then there is  $\epsilon > 0$  such that

$$\|(1 - \varphi_j)a_j\|_{C^0((D_{r_{j-1}}^j)^c)} \geq \epsilon$$

should hold for all large  $j \gg 1$ .

For each  $f \in C_0(\mathbb{R})$  and  $\epsilon > 0$ , there is some  $r > 0$  such that  $\beta(f) \in S\mathfrak{C}(H)$  satisfies the estimate  $\|\beta(f)\|_{C^0((D_r)^c)} < \epsilon$ . So  $a \in S\mathfrak{C}(H)$  also satisfies the estimate:

$$\|a\|_{C^0((D_r)^c)} < \epsilon$$

for all large  $r \gg 1$ . Since  $\beta(a_j)$  converges to  $a$  in  $C^0$  norm, this contradicts to the above lower bound.  $\square$

**6.3. Asymptotic unitary operators.** Let  $l : H' \cong H$  be a linear isomorphism between Hilbert spaces, and let :

$$\bar{l} = l \circ \sqrt{l^* \circ l}^{-1} : H' \rightarrow H$$

be the unitary corresponding to the polar decomposition of  $l$ .

For any finite dimensional linear subspace  $V \subset H$ , let us compare:

$$V' \equiv l^{-1}(V), \quad \bar{V}' \equiv \bar{l}^{-1}(V)$$

in  $H'$ .

Let us put  $l(V'_1, V'_2) = \sup_{v_2} \inf_{v_1} \{ \|v_1 - v_2\| : \|v_1\| = \|v_2\| = 1, v_i \in V'_i \}$ .  $l(V'_1; V'_2) = 0$  holds if and only if  $V'_1$  contains  $V'_2$ . Then we introduce the distance between planes by:

$$d(V'_1, V'_2) = \inf \{ l(V'_1; V'_2), l(V'_2; V'_1) \}.$$

**Lemma 6.4.** *Let  $W'_i$  be any finite dimensional linear exhaustion  $W'_i$  with  $W'_i \subset W'_{i+1}$  so that  $\cup_i W'_i \subset H'$  is dense.*

*For any finite dimensional linear subspace  $V' \subset H'$  and any small  $\epsilon > 0$ , there is some  $i_0$  such that for all  $i \geq i_0$ , the canonical map:*

$$Cl(\bar{V}') \hat{\otimes} Cl(\bar{U}'_i) \rightarrow Cl(\bar{W}'_i)$$

*is a linear isomorphism and is  $\epsilon$  \* homomorphism, where  $U'_i \subset W'_i$  is another linear subspace.*

*Proof. Step 1:* We claim that for any finite dimensional linear subspace  $V' \subset H'$  and any  $\epsilon > 0$ , the estimates:

$$d(\bar{V}', \bar{W}'_i) < \epsilon$$

holds. Actually  $\bar{W}' = H'$  holds when  $W' = H'$  since the polar decomposition gives the unitary. So for any finite dimensional linear exhaustion  $W'_i$  with  $\cup_i W'_i \subset H'$  is dense, the same thing holds for another

exhaustion  $\{\bar{W}_i\}_i$  so that  $\cup_i \bar{W}'_i \subset H'$  is dense, where  $\bar{W}'_i = \bar{l}^{-1}(l(W'_i))$ . In particular the claim follows.

**Step 2:** Let  $\text{pr}_i : H' \rightarrow \bar{W}'_i$  be the orthogonal projection. Then the operator norm  $\|(1 - \text{pr}_i)|\bar{V}'\| < \epsilon$  is bounded by  $\epsilon$  and hence its orthogonal complement  $\bar{U}'_i \equiv \text{pr}_i(\bar{V}')^\perp \subset \bar{W}'_i$  admits the direct sum  $\bar{V}' \oplus \bar{U}'_i$ . It is ‘almost’ orthogonal in the sense that any  $\bar{V}' \in v'$  and  $u' \in \bar{U}'_i$  with  $\|v'\| = 1$  and  $\|u'\| = 1$  admits the estimate:

$$|\langle v', u' \rangle| < \epsilon.$$

This implies the conclusion.  $\square$

**Remark 6.5.** Let  $U' = \sqrt{l^* \circ \bar{l}} \circ l^{-1}(\bar{U}') \subset W'$ . Even though the direct sum  $V' \oplus U' = W'$  holds, this sum is far from orthogonal one in general.

**Definition 6.2.** Let  $H_i$  be Hilbert spaces and  $l : H_1 \cong H_2$  be a linear isomorphism.

$l$  is asymptotically unitary, if for any  $\epsilon > 0$ , there is a finite dimensional linear subspace  $V \subset H_1$  such that the restriction:

$$l : V^\perp \cong l(V^\perp)$$

satisfies the estimate:

$$\|(l - \bar{l})|V^\perp\| < \epsilon$$

on its operator norm, where  $\bar{l}$  is the unitary of the polar decomposition of  $l$ .

Let  $l : H' \cong H$  be a linear isomorphism between the Sobolev spaces, given by an elliptic operator over  $X$ . Recall their Sobolev norms which have been introduced in section 3. We denote by  $P$  the spectral projection of  $l^* \circ l$ .

**Lemma 6.6.** The operator:

$$l : H' \cong H$$

satisfies a property that for any  $\epsilon > 0$ , there is  $\lambda_0 \gg 1$  such that the operator norm of the restriction of  $l^* \circ l$  on  $P[\lambda_0, \infty) \subset H'$  satisfy the estimate:

$$\|(l - \bar{l})|P[\lambda_0, \infty)\| < \epsilon$$

where  $\bar{\phantom{x}}$  is the unitary of the polar decomposition.

In particular the operator:

$$U \equiv \bar{l}^{-1} \circ l = \sqrt{l^* l} : H' \cong H'$$



satisfies the estimate:

$$||(U - id)|P[\lambda_0, \infty)|| < \epsilon.$$

*Proof.* The latter statement follows from the former.

Let us verify the former property. Let us put:

$$P_N(c) = \frac{c^N - 1}{c - 1}$$

for  $c > 1$ . Notice the equalities  $cP_{N-1}(c) + 1 = P_N(c)$ .

If  $u$  is an eigenvector vector with  $l^*l(u) = \lambda^2 u$ , then the formulas  $||u||_{L_k^2}^2 = P_k(\lambda^2)||u||_{L^2}^2$  hold for all  $k \geq 0$ . One can check this by induction as follows:

$$\begin{aligned} \langle u, u \rangle_{L_k^2} &= \langle l(u), l(u) \rangle_{L_{k-1}^2} + \langle u, u \rangle_{L^2} \\ &= \langle l^*l(u), u \rangle_{L_{k-1}^2} + \langle u, u \rangle_{L^2} = \lambda^2 \langle u, u \rangle_{L_{k-1}^2} + ||u||_{L^2}^2 \end{aligned}$$

In particular if  $u \in L_k^2(X; E)$  with  $||u||_{L_k^2} = 1$  lies in the image of spectral projection to  $[\lambda_0^2, \infty)$  on  $l^* \circ l$ , then  $||u||_{L^2}$  is sufficiently small for large  $\lambda_0 >> 1$ . So it follows from the formula:

$$\langle u, u \rangle_{L_k^2} = \langle l(u), l(u) \rangle_{L_{k-1}^2} + \langle u, u \rangle_{L^2}$$

that  $l$  is close to preserve the norms. □

Let  $l : L_k^2(X) \cong L_{k-1}^2(X)$  be an elliptic operator which gives an isomorphism.

Let  $K \subset X$  be a compact subset and consider the restriction:

$$l : L_k^2(K)_0 \rightarrow L_{k-1}^2(K)_0.$$

**Proposition 6.7.** *The restriction  $l : L_k^2(K)_0 \rightarrow L_{k-1}^2(K)_0$  is asymptotically unitary.*

*In particular  $U \equiv \bar{l}^{-1} \circ l$  is asymptotic identity.*

*Proof. Step 1:* We verify that for any  $\epsilon > 0$ , there is a finite dimensional linear subspace  $V \subset L_k^2(K)_0$  such that the restriction:

$$l : V^\perp \cap L_k^2(K)_0 \rightarrow L_{k-1}^2(K)_0$$

satisfies the estimate:

$$||(l - \bar{l})|V^\perp \cap L_k^2(K)_0|| < \epsilon.$$

In particular we obtain the estimate:

$$||(U - id)|V^\perp \cap L_k^2(K)_0|| < \epsilon.$$

Notice that it can happen that an eigenvalue has infinite multiplicity on  $L^2(X)$ , in which case the above estimate fails over  $X$ . Such situation can happen only when  $X$  is non compact.

**Step 2:** Let  $P[0, \lambda] : L^2(X) \rightarrow L^2(X)$  be the spectral projection of  $l^* \circ l$ , and  $B_K \subset L_k^2(K; E)_0$  be the unit ball.

We claim that  $P[0, \lambda](B_K) \subset L_k^2(X; E)$  is relatively compact for every  $\lambda > 0$ . Take any sequence  $\{u_i\}_i \subset L_k^2(K; E)_0$  with  $\|u_i\|_{L_k^2} = 1$ , and put  $u_i^1 = P[0, \lambda](u_i)$  with  $u_i^2 = u_i - u_i^1 \in P(\lambda, \infty)(B_K)$ .

Notice the formula  $\langle u_i, u_j \rangle_{L_{k-1}^2} = \langle u_i^1, u_j^1 \rangle_{L_{k-1}^2} + \langle u_i^2, u_j^2 \rangle_{L_{k-1}^2}$ , and hence:

$$\|u_i - u_j\|_{L_{k-1}^2}^2 = \|u_i^1 - u_j^1\|_{L_{k-1}^2}^2 + \|u_i^2 - u_j^2\|_{L_{k-1}^2}^2.$$

Since  $\{u_i\}_i$  has compact support, a subsequence converge in  $L_{k-1}^2(K; E)_0$ . So one may assume that the left hand side converge to 0. In particular  $\lim_{i \rightarrow \infty} \|u_i^1 - u_j^1\|_{L_{k-1}^2} = 0$  holds. Since the bound:

$$\|u_i^1 - u_j^1\|_{L_k^2}^2 \leq C \|u_i^1 - u_j^1\|_{L_{k-1}^2}^2 \rightarrow 0$$

holds for some constant  $C$ ,  $\|u_i^1\|_{L_{k-1}^2}^2 \rightarrow 0$  as  $i \rightarrow \infty$  holds.

This verifies the claim.

**Step 3:** Let us take an orthonormal basis  $\{u_i\}_i \subset L_k^2(K; E)_0$ . It follows from step 2 that a subsequence of  $\{u_i^1\}_i$  converge in  $L_k^2$ . In particular for any  $\epsilon > 0$ , there is a finite dimensional vector space  $V'$  spanned by  $\{u_{i_1}^1, \dots, u_{i_m}^1\}$  for some  $\{i_1, \dots, i_m\}$  such that

$$\|(1 - \text{pr}_{V'})u_i^1\|_{L_k^2} < \epsilon$$

hold for all  $i$ .

Let  $V \subset L_k^2(K; E)_0$  be a finite dimensional vector space spanned by  $\{u_{i_1}, \dots, u_{i_m}\}$ . Then the inclusion:

$$V \subset V' \oplus P[\lambda, \infty)$$

holds. Moreover, for any  $i$ , there is  $u'_i \in L_k^2(X; E)$  with  $\|u_i^1 - (u'_i)^1\|_{L_k^2} < \epsilon$  such that

$$u'_i \in V' \oplus P[\lambda, \infty)$$

hold. Then the conclusion follows by lemma 6.6. □

**Lemma 6.8.** *Let  $l : H' \cong H$  be asymptotically unitary. For any  $\epsilon > 0$ , there is a finite dimensional vector subspace  $V'_0 \subset H'$  such that the estimates:*

$$d(V', (\bar{l}^* \circ l)(V')) < \epsilon$$

*hold for any  $V' \supset V_0$ .*

*Proof.* Choose a finite dimensional vector space  $V' \subset H'$  with:

$$\|(\bar{l}^* \circ l) - \text{id}\|(V')^\perp < \epsilon.$$

Suppose:

$$d(V', (\bar{l}^* \circ l)((V')^\perp)) \geq \epsilon_0 > 0$$

could hold. Take a unit vector  $u' \in V'$  with  $\|\text{pr}(u')\| \geq \epsilon_0$ , where  $\text{pr} : H' \rightarrow (\bar{l}^* \circ l)((V')^\perp)$  is the orthogonal projection. Put:

$$u' = (\bar{l}^* \circ l)(v'_1) + (\bar{l}^* \circ l)(v'_2) \equiv u'_1 + u'_2$$

with  $u'_2 = \text{pr}(u')$  and  $v'_2 \in (V')^\perp$  with  $C \geq \|v'_2\| \geq C\epsilon_0$ .

Then we have the estimate:

$$\begin{aligned} 1 &\leq \|u'\|^2 + \|v'_2\|^2 = \|u' - v'_2\|^2 \\ &\leq (\|u'_1\| + \epsilon\|v'_2\|)^2 \leq (1 - \epsilon_0 + \epsilon\|v'_2\|)^2 \end{aligned}$$

The last term is bounded by  $(1 - \frac{\epsilon_0}{2})^2$ , if we choose sufficiently small  $\epsilon > 0$ . This is a contradiction.  $\square$

**Remark 6.9.** *In the case of the covering monopole map:*

$$\begin{aligned} \tilde{\mu} : L_k^2((X, g); \tilde{S}^+) \oplus L_k^2((X, g); \Lambda^1 \otimes i\mathbb{R}) \cap \text{Ker } d^* \rightarrow \\ L_{k-1}^2((X, g); \tilde{S}^- \oplus \Lambda_+^2 \otimes i\mathbb{R}) \oplus H^1(X) \\ (\phi, a) \rightarrow (F_{\tilde{A}_0, \tilde{\psi}_0}(\psi, a), [a]) \end{aligned}$$

the target space is the sum of the Sobolev space with  $H^1$ , where the latter space is infinite dimensional if not zero.

By Hodge theory,  $\text{Ker } d^*$  decomposes as  $d^*(L_{k+1}^2(X; \Lambda_+^2)) \oplus H^1(X)$ , and so:

$$(\cup_i d^*(L_{k+1}^2(K_i; \Lambda_+^2)_0)) \oplus H^1(X)$$

is dense in  $\text{Ker } d^*$ .

The restriction of the linearized map on the harmonic part is in fact isometry. Hence the covering monopole map is also asymptotically unitary over compactly supported Sobolev spaces.

**6.4. A variant of Hilbert Clifford algebra.** Let us introduce a variant of the HKT construction. Actually it turns out that the result of the  $C^*$  algebra coincides with the original one given by HKT.

Let  $l : H' \cong H$  be an asymptotically unitary isomorphism. Let  $E \subset H$  be a finite dimensional Euclidean space, and  $E' = l^{-1}(E)$  and  $\bar{E}' = \bar{l}^{-1}(E)$  be as above. The map:

$$C_l \equiv \bar{l}^{-1} \circ l : E' \rightarrow \bar{E}' \hookrightarrow Cl(\bar{E}')$$

is also called the Clifford operator. Let us denote:

$$\mathfrak{C}_l(E) = C_0(E', Cl(\bar{E}'))$$

and introduce a  $*$ -homomorphism:

$$\beta : C_0(\mathbb{R}) \rightarrow S\mathfrak{C}_l(E) \equiv C_0(\mathbb{R}) \hat{\otimes} \mathfrak{C}_l(E)$$

defined by  $\beta : f \rightarrow f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_l)$  by functional calculus.

Let  $E'_a \subset E'_b \subset H'$  be a pair of finite dimensional linear subspaces, and denote the orthogonal complement as  $E'_{ba} = E'_b \cap (E'_a)^\perp$ .

**Lemma 6.10.** *Let  $l : H' \cong H$  be asymptotically unitary. For any  $\epsilon > 0$ , there is a finite dimensional vector space  $V' \subset H'$  such that there is a canonical linear isomorphism:*

$$S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b)$$

which is ‘almost’  $*$ -homomorphism if the inclusion  $V' \subset E'_a$  holds.

*Proof.* Let  $V'$  be the vector subspace in lemma 6.8. Let  $\hat{E}'_{ba} \subset \bar{E}'_b$  be the orthogonal complement of  $\bar{E}'_a$ , and consider the orthogonal projection:

$$\text{pr} : \bar{E}'_{ba} \rightarrow \hat{E}'_{ba}.$$

$l^* \circ l$  is almost unitary on  $E_{ba}$  so that the operator norm satisfies the estimate  $\|(l^* \circ l) - \text{id}|_{E_{ba}}\| < \epsilon$ . The estimate  $d(E_a, \bar{E}_a) < \epsilon$  also follows from lemma 6.8. In particular the operator norm of the above projection satisfies:

$$\|\text{pr} - \text{id}|_{\bar{E}'_{ba}}\| < 2\epsilon.$$

Let  $\bar{\text{pr}} : \bar{E}'_{ba} \rightarrow \hat{E}'_{ba}$  be the unitary of the polar decomposition. It also satisfies the estimate  $\|\bar{\text{pr}} - \text{id}|_{\bar{E}'_{ba}}\| < 4\epsilon$ , which induces a  $*$ -isomorphism:

$$\bar{\text{pr}} : Cl(\bar{E}'_{ba}) \cong Cl(\hat{E}'_{ba}).$$

It extends to the  $*$ -isomorphism:

$$\bar{\text{pr}} \hat{\otimes} 1 : Cl(\bar{E}'_{ba}) \hat{\otimes} Cl(\bar{E}'_a) \cong Cl(\hat{E}'_{ba}) \hat{\otimes} Cl(\bar{E}'_a) \cong Cl(\bar{E}'_b)$$

which induces the desired  $*$ -isomorphism:

$$S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b).$$

□

**Remark 6.11.** *Surely  $\text{pr}$  induces a linear map:*

$$\text{pr} : Cl(\bar{E}'_{ba}) \rightarrow Cl(\hat{E}'_{ba})$$

by letting  $u = u_1 + u_2 \in Cl(\hat{E}'_{ba}) \oplus Cl^0(\bar{E}'_a)$  to  $u_1 \in Cl(\hat{E}'_{ba})$ , where  $Cl^0(E)$  is the scalar less part of  $Cl(E)$ .

However it cannot be ‘almost’  $*$ -isomorphic in general, as  $\dim E'_{ba}$  grow. To see this, let us take any  $u' \in \bar{E}'_{ba}$  and put  $u''' = u' - \text{pr}(u') \equiv u' - u''$ . For any orthonormal basis  $\{u'_1, u'_2, \dots\}$  of  $\bar{E}'_{ba}$ , consider their product  $u'_1 u'_2 \dots u'_m \in Cl(\bar{E}'_{ba})$ .

$$\begin{aligned} u'_1 u'_2 \dots &= (u''_1 + u'''_1)(u''_2 + u'''_2)(u''_3 + u'''_3) \dots (u''_m + u'''_m) \\ &= u''_1 u''_2 \dots u''_m + \text{other terms} \end{aligned}$$

Each norm  $\|u''_i\| < 1$  is strictly less than 1, and hence the norm of their product in the first term above may degenerate to zero.

There is a canonical map:

$$\beta_{ba} = \beta \hat{\otimes} 1 : S\mathfrak{C}_l(E'_a) \rightarrow S\mathfrak{C}_l(E'_{ba}) \hat{\otimes} \mathfrak{C}_l(E'_a) \cong S\mathfrak{C}_l(E'_b)$$

by the same way as the Bott map, just by replacing  $C$  by  $C_l$ . Then the limit:

$$\|u\| \equiv \lim_{\dim E'_b \rightarrow \infty} \|\beta_{ba}(u)\|_{S\mathfrak{C}_l(E'_b)}$$

exists, which gives a norm on  $S\mathfrak{C}_l(E'_a)$ .

**Definition 6.3.** The direct limit  $C^*$  algebra is given by:

$$S\mathfrak{C}_l(H') = \lim_a S\mathfrak{C}_l(E'_a)$$

where the norm is equipped with the above one.

**Lemma 6.12.** Assume that  $l$  is asymptotically unitary.

Then there is a canonical  $*$ -isomorphism:

$$S\mathfrak{C}_l(H') \rightarrow S\mathfrak{C}(H')$$

between the two Clifford  $C^*$  algebras.

*Proof. Step 1:* It follows from lemma 6.8 that for any  $\epsilon > 0$ , there is a finite dimensional vector space  $V'_0 \subset H'$  such that for all  $V' \supset V'_0$ ,  $d(V', \bar{l}^* \circ l(V')) < \epsilon$  holds. By definition,  $d((V')^\perp, \bar{l}^* \circ l((V')^\perp)) < \epsilon$  also holds.

Let  $\text{pr}_1 : \bar{V}' \cong V'$  and  $\text{pr}_2 : (\bar{V}')^\perp \cong (V')^\perp$  be the orthogonal projections and  $\bar{\text{pr}}_i$  be their corresponding unitaries. Both satisfy the bounds:

$$\|\bar{\text{pr}}_i - \text{id}\| < 2\epsilon.$$

They extend to the  $*$ -algebras isomorphisms:

$$\begin{aligned} \bar{\text{pr}}_1 : Cl(\bar{V}') &\cong Cl(V'), \\ \bar{\text{pr}}_2 : Cl((\bar{V}')^\perp) &\cong Cl((V')^\perp). \end{aligned}$$

In particular the former induces the  $*$ -isomorphism:

$$\bar{\text{pr}}_1 : C_0(V', Cl(\bar{V}')) \cong C_0(V', Cl(V')).$$

**Step 2:** Let us take two Bott maps:

$$\begin{aligned}\beta_1 : C_0(\mathbb{R}) &\rightarrow S\mathfrak{C}_l(W'), & \beta_1(f) &= f(X \hat{\otimes} 1 + 1 \hat{\otimes} C_l), \\ \beta_2 : C_0(\mathbb{R}) &\rightarrow S\mathfrak{C}_l(W'), & \beta_2(f) &= f(X \hat{\otimes} 1 + 1 \hat{\otimes} C)\end{aligned}$$

and consider the diagram:

$$\begin{array}{ccc} S\mathfrak{C}_l(V') & \xrightarrow{\beta_1} & S\mathfrak{C}_l(W') \hat{\otimes}_0 (V', Cl(\bar{V}')) \\ \bar{\text{pr}}_1 \downarrow & & \bar{\text{pr}} \downarrow \\ S\mathfrak{C}(V') & \xrightarrow{\beta_2} & S\mathfrak{C}(W') \hat{\otimes} C_0(V', Cl(V')) \end{array}$$

It follows from the estimate:

$$\|\bar{\text{pr}}_2(\bar{l}^* \circ l(u)) - u\| \leq 3\epsilon \|u\|$$

for any  $u' \in (V')^\perp$  that this diagram satisfies the estimate:

$$\|\bar{\text{pr}} \circ \beta_1(a) - \beta_2 \circ \bar{\text{pr}}_1(a)\| < 4\epsilon.$$

$\epsilon$  can be arbitrarily small by choosing large  $V'$ .

**Step 3:** Let us take an element  $a \in S\mathfrak{C}_l(H')$ , and choose  $a_i \in S\mathfrak{C}_l(V'_i)$  with  $\lim_i \|\beta_1(a_i) - a\| = 0$ , where  $\beta_1(a_i) \in S\mathfrak{C}_l(H')$  and  $V' \subset V'_i \subset V'_{i+1} \subset H'$  is an increasing sequence of finite dimensional vector spaces with dense union.

It follows from the above estimate on the diagram that

$$\begin{aligned}\bar{\text{pr}} : S\mathfrak{C}_l(H') &\rightarrow S\mathfrak{C}(H'), \\ \bar{\text{pr}}(a) &= \lim_i \beta_2(\bar{\text{pr}}(a_i))\end{aligned}$$

is uniquely defined and independent of choice of  $a_i$ .

It is easy to check that this assignment gives a  $*$ -homomorphism. To see that it is isomorphic, one can consider another projections to the converse way, from  $\text{pr}' : V' \cong \bar{V}'$ . A parallel argument gives another  $*$ -homomorphism  $\bar{\text{pr}}' : S\mathfrak{C}(H') \rightarrow S\mathfrak{C}_l(H')$ , and their compositions both give the identities.

□

## 7. HIGHER DEGREE $*$ HOMOMORPHISM

**7.1. Degree of proper maps.** Let  $E', E$  be finite dimensional vector spaces, and  $F = l + c : E' \rightarrow E$  be a proper smooth map whose linear part  $l : E' \cong E$  gives an isomorphism.

Let us reconstruct the degree of  $F \in \mathbb{Z}$  by use of  $l$ . Let  $\bar{l} : E' \rightarrow E$  be the unitary corresponding to the polar decomposition.  $\bar{l}$  induces the algebras isomorphisms:

$$\bar{l} : Cl(E') \cong Cl(E)$$

Then we have the induced  $*$  homomorphism:

$$\begin{aligned} F^* : C_0(E, Cl(E)) &\rightarrow C_0(E', Cl(E')) \\ F^*(h)(v) &= \bar{l}^{-1}(h(F(v))). \end{aligned}$$

Let us consider the induced homomorphism on  $K$  groups:

$$\begin{array}{ccc} F^* : K(S\mathfrak{C}(E)) & \rightarrow & K(S\mathfrak{C}(E')) \\ \uparrow \beta & & \uparrow \beta \\ \tilde{F}^* : K(C_0(\mathbb{R})) & \rightarrow & K(C_0(\mathbb{R})) \end{array}$$

where  $\tilde{F}^*$  is determined so that the diagram commutes.

Let us equip with orientations on both  $E'$  and  $E$  so that  $l$  preserves their orientations.

**Lemma 7.1.** *Passing through the isomorphism  $K(C_0(\mathbb{R})) \cong \mathbb{Z}$ ,*

$$\tilde{F}^* : \mathbb{Z} \rightarrow \mathbb{Z}$$

*is given by multiplication by the degree of  $F$ .*

*Proof. Step 1:* Let us consider the composition of  $*$  homomorphisms:

$$C_0(E, Cl(E)) \rightarrow C_0(E', Cl(E')) \cong C_0(E, Cl(E))$$

where the first map is  $F^*$  and the second one is given by:

$$(\bar{l}^{-1})^*(h')(v) \equiv \bar{l}(h'(\bar{l}^{-1}(v))).$$

The latter gives an isomorphism since  $l$  is isomorphic. So it is enough to see the conclusion for the composition. The composition is given by:

$$h \rightarrow \{ v \rightarrow h(F \circ \bar{l}^{-1}(v)) \}$$

**Step 2:** Let  $l_t : E' \cong E$  be another family of linear isomorphisms with  $l_0 = \bar{l}$  and  $l_1 = l$ . It induces a family of  $*$ -homomorphisms:

$$\begin{aligned} F_t^* : C_0(E, Cl(E)) &\rightarrow C_0(E, Cl(E)) \\ h &\rightarrow \{ v \rightarrow h(F \circ (l_t)^{-1}(v)) \}. \end{aligned}$$

Since homotopic  $*$ -homomorphisms induce the same maps between their  $K$ -group, it is enough to see the conclusion for  $F_1^*$ . Notice the equality  $F \circ l^{-1} = 1 + c \circ l^{-1}$ , and hence it is enough to assume  $l$  is the identity.

**Step 3:** When  $l$  is the identity,  $F^* : S\mathfrak{C}(E) \rightarrow S\mathfrak{C}(E)$  is given by:

$$\text{id} \times F^* : S\mathfrak{C}(E) \cong (S \otimes Cl(E)) \otimes C_0(E) \rightarrow (S \otimes Cl(E)) \otimes C_0(E)$$

whose induced homomorphism on  $K$  group is given by degree  $F$ , passing through the isomorphism:

$$K_1(S \otimes Cl(E) \otimes C_0(E)) \cong K_*(C_0(E)) = K^*(E) \cong \mathbb{Z}$$

where  $*$  is 0 or 1 with respect to  $\dim E$  is even or odd.  $\square$

**7.2. Approximation of maps.** Let  $F = l + c : L_{k+1}^2(X) \rightarrow L_k^2(X)$  be a non linear map such that  $l$  is elliptic and  $c$  is locally compact.

Suppose:

$$l : L_{k+1}^2(X) \cong L_k^2(X)$$

gives an isomorphism, and for each closed linear vector subspace  $V \subset L_{k+1}^2(X)$ , denote the orthogonal projection by:

$$\text{pr}_V : L_k^2(X) \rightarrow l(V).$$

Then we have the modified maps:

$$F_V = l + \text{pr}_V \circ c : V \rightarrow l(V)$$

There is a canonical map  $F_V \rightarrow F_{V'}$  if the inclusion  $V' \subset V$  holds.

Choose exhaustion  $K_0 \subset \subset K_1 \subset \subset K_2 \subset \subset \cdots \subset X$ . The union  $\cup_i L_k^2(K_i)_0 \subset L_k^2(X)$  is dense. Fix  $m$  and choose approximation by finite dimensional linear subspaces:

$$V_0 \subset \cdots \subset V_i \subset V_{i+1} \subset \cdots \subset L_{k+1}^2(K_m)_0.$$

It follows from lemma 5.1(2) that the family  $\{F_{V_i}\}_i$  converge to the non linear map:

$$F_m = l + \text{pr}_m \circ c : L_{k+1}^2(K_m)_0 \rightarrow L_k^2(K_m)_0$$

which is uniquely defined independently of choice of approximations, where  $\text{pr}_m : L_k^2(X) \rightarrow L_k^2(K_m)_0$  is the orthogonal projection. In fact for each  $v \in L_{k+1}^2(K_m)_0$ , let us choose any approximation  $v_i \in V_i$  which converge to  $v$  in  $L_{k+1}^2(K_m)_0$ . Then let us check that  $\lim_i F_{V_i}(v_i)$  surely exists. For  $i \leq j$ ,

$$\begin{aligned} \|F_{V_i}(v_i) - F_{V_j}(v_j)\| &\leq \|l(v_i) - l(v_j)\| + \|\text{pr}_i c(v_i) - \text{pr}_j c(v_j)\| \\ &\leq \|l(v_i) - l(v_j)\| + \|\text{pr}_i c(v_i) - \text{pr}_i c(v_j)\| + \|(\text{pr}_i - \text{pr}_j)c(v_j)\| \\ &\leq \|l(v_i) - l(v_j)\| + \|c(v_i) - c(v_j)\| + \|(\text{pr}_i - \text{pr}_j)c(v_j)\| \end{aligned}$$

which converge to zero, since in the right hand side, the first and the second terms both converge to zero. The third one also converges to zero since the operators  $\text{pr}_i - \text{pr}_j$  weakly converge to zero.



**7.3. Induced map.** Let  $D_r^m \subset L_{k+1}^2(K_m)_0$  be  $r$ -balls. Let us put  $H'_m = L_{k+1}^2(K_m)_0$  and  $H_m = L_k^2(K_m)_0$ .

**Proposition 7.2.** *Let  $F = l + c : H' \rightarrow H$  be a  $\Gamma$  equivariant map which is locally strongly proper, and whose linear part  $l : H' \cong H$  gives an isomorphism.*

*Then there exists the induced equivariant  $*$ -homomorphism:*

$$F^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}(H').$$

*Proof. Step 1:* We shall define  $F^*(a)$  for an element  $a \in \beta(S\mathfrak{C}(V)) \subset S\mathfrak{C}(H)$  and finite dimensional linear subspace  $V \subset H$ . Such elements consist of a dense subset in  $S\mathfrak{C}(H)$ , and it is enough to associate the corresponding elements  $F^*(a)$  for them, by the following abstract property.

It is a basic fact in theory of  $C^*$  algebras that densely defined  $*$ -homomorphism between  $C^*$  algebras extends to the norm decreasing  $*$ -homomorphism between them. For convenience, we describe a proof of this fact. Let  $A$  and  $B$  be two  $C^*$  algebras, and  $\varphi$  be a densely defined  $*$ -homomorphism from  $A$  to  $B$ . Let  $r$  denote the spectral radius. The inequality  $r(a) \geq r(\varphi(a))$  follows easily from the definition. Then the estimate holds:

$$\begin{aligned} \|\varphi(a)\|^2 &= \|\varphi(a)^*\varphi(a)\| = \|\varphi(a^*a)\| = r(\varphi(a^*a)) \\ &\leq r(a^*a) = \|a^*a\| = \|a\|^2. \end{aligned}$$

**Step 2:** Since  $K_m$  are compact, the images  $c(D_r^m) \subset L_k^2(X)$  are relatively compact. There exist finite dimensional linear subspaces  $V_r^m \subset L_{k+1}^2(K_m)_0$  and positive numbers  $\epsilon_r^m > 0$  with  $\epsilon_r^m \rightarrow 0$  such that  $\epsilon_r^m$  neighborhoods of  $l(V_r^m)$  contain  $\text{pr}_m(c(D_r^m))$  in  $L_k^2(K_m)_0$ :

$$L_k^2(K_m)_0 \supset N_{\epsilon_r^m}(V_r^m) \supset \text{pr}_m(c(D_r^m))$$

where  $\text{pr}_m : L_k^2(X) \rightarrow L_k^2(K_m)_0$  are the orthogonal projections. One may assume the embeddings for all  $r$  and  $m$ :

$$\begin{array}{ccccc} V_r^m & \subset & V_r^{m+1} & \subset & D_r \\ \cap & & \cap & & \cap \\ V_{r+1}^m & \subset & V_{r+1}^{m+1} & \subset & D_{r+1} \\ \cap & & \cap & & \cap \end{array}$$

$$L_{k+1}^2(K_m)_0 \subset L_{k+1}^2(K_{m+1})_0 \subset L_{k+1}^2(X)$$

**Step 3:** Suppose  $l$  is unitary. Then  $S\mathfrak{C}_l(H)$  coincides with  $S\mathfrak{C}(H)$  by definition.

Firstly we construct the induced map:

$$F^* : S\mathfrak{C}(H_m) \rightarrow S\mathfrak{C}(H'_m)$$

as follows.

Let  $\beta : C_0(\mathbb{R}) \rightarrow S\mathfrak{C}(W)$  be the Bott map. It follows from the construction that  $\beta(f) \in S\mathfrak{C}(D_r)$  holds if  $f \in C_c^\infty(-r, r)$ , where  $D_r \subset W$  is  $r$ -ball.

**Step 4:** Take  $\alpha = [f \hat{\otimes} a] \in S\mathfrak{C}(H_m)$ , where  $[\ ]$  implies the image of the Bott map  $\beta : S\mathfrak{C}(W) \rightarrow S\mathfrak{C}(H_m)$  for some finite dimensional linear subspace  $W$ . A general element  $\alpha$  is obtained as limit of the form  $\alpha = \lim_i [\alpha_i]$  for some  $\alpha_i \in S\mathfrak{C}(W_i)$  and  $W_i$ .

Firstly suppose  $f \in C_c(\mathbb{R})$  and  $a \in C_c(W; Cl(W))$  for some  $W \subset L_k^2(K_m)_0$ . One may assume  $f \in C_c(-r, r)$  and  $a \in C_c(D_r^m; Cl(W_r^m))$ , where  $W_r^m = l(V_r^m)$ .

Put  $\text{pr}^r = \text{pr}_{W_r^m}$  and  $F_r = l + \text{pr}^r \circ c$ . Then it follows from proposition 5.10 (see remark 5.6) that one obtains the induced map:

$$\begin{aligned} F_r^* : S\mathfrak{C}(W_r^m) &\rightarrow S\mathfrak{C}(V_{r'}^m) \\ F_r^*(f \hat{\otimes} a)(v) &= f \hat{\otimes} l^{-1}(a(F_r(v))) \end{aligned}$$

Let us put  $W_{r,r'}^m = W_{r'}^m \cap (W_r^m)^\perp$ , and  $\beta(f) \in S\mathfrak{C}(W_{r,r+1}^m)$  be the image of the Bott map. It lies in  $S\mathfrak{C}(D_r \cap W_{r,r+1}^m)$  by step 3. Then we have the diagram below:

$$\begin{array}{ccc} F_r^* : & S_r \hat{\otimes} C_c(D_r^m; Cl(W_r^m)) & \rightarrow & S_r \hat{\otimes} C_c(D_{r'}^m; Cl(V_{r'}^m)) \\ & \downarrow \beta & & \downarrow \beta \\ F_{r+1}^* : & S_{r+1} \hat{\otimes} C_c(D_{r+1}^m; Cl(W_{r+1}^m)) & \rightarrow & S_{r+1} \hat{\otimes} C_c(D_{(r+1)'}^m; Cl(V_{r+1}^m)) \end{array}$$

where  $S_r \equiv C_c(-r, r)$ .

This diagram does not commute in the presence of the compact non linear term in  $F_r$ . But error from commutativity is bounded by the constants  $\epsilon_r^m > 0$ . In particular we have the convergence on the norm of the  $*$ -homomorphisms:

$$\lim_{r \rightarrow \infty} \|(\beta \circ F_r^* - F_{r+1}^* \circ \beta) | S_r \hat{\otimes} C_c(D_r^m; Cl(W_r^m))\| = 0$$

**Step 5:** Suppose  $f \in C_0(\mathbb{R})$  and  $a \in C_0(W; Cl(W))$ . Then there are approximations  $f_r \in C_c(-r, r)$  and  $a_r \in C_c(D_r^m; Cl(W_r^m))$ , where both converge to  $f$  and  $a$  in operator norms respectively. It follows from step 4 that the left hand side:

$$F^*(f \hat{\otimes} a) \equiv \lim_{r \rightarrow \infty} F_r^*(f_r \hat{\otimes} a_r) = F^*(\beta(f \hat{\otimes} a))$$

is formulated by the middle one and is well defined, where  $\beta(f \hat{\otimes} a) \in S\mathfrak{C}(W_{r+1}^m)$ . This is the desired  $*$ -homomorphism:

$$F^* : S\mathfrak{C}(H_m) \rightarrow S\mathfrak{C}(H'_m).$$

**Step 6:** Let us consider the general case. Let  $W \subset H = L_k^2(X)$  and take an element  $f \hat{\otimes} a \in S\mathfrak{C}(W)$ . In general it can be approximated as:

$$f \hat{\otimes} a = \lim_{m \rightarrow \infty} f_m \hat{\otimes} a_m \in S\mathfrak{C}(W_m)$$

for  $W_m \subset L_k^2(K_m)_0$ . Then we define the induced  $*$ -homomorphism:

$$F^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}(H')$$

by:

$$F^*(f \hat{\otimes} a) = \lim_{m \rightarrow \infty} F^*(f_m \hat{\otimes} a_m) \in S\mathfrak{C}(H')$$

where we regard  $S\mathfrak{C}(H'_m) \subset S\mathfrak{C}(H')$  passing through another Bott map:

$$\beta : S\mathfrak{C}(H'_m) \rightarrow S\mathfrak{C}(H') = S\mathfrak{C}(H'_m \oplus (H'_m)^\perp).$$

**Step 7:** Suppose  $l$  is asymptotically unitary rather than unitary. Then it follows lemma 6.12 that both  $S\mathfrak{C}_l(H)$  and  $S\mathfrak{C}(H)$  are isomorphic.

We follow the construction in step 4  $\sim$  6 by replacing  $S\mathfrak{C}(H')$  by  $S\mathfrak{C}_l(H')$ . In particular we replace the induced map by the following:

$$\begin{aligned} F_r^* : S\mathfrak{C}(W_r^m) &\rightarrow S\mathfrak{C}_l(V_r^m) \\ F_r^*(f \hat{\otimes} a)(v) &= f \hat{\otimes} \bar{l}^{-1}(a(F_r(v))) \end{aligned}$$

where  $\bar{l}$  is the unitary in the polar decomposition of  $l$ .

Parallel argument to step 4 and 5 with the proof in lemma 6.12 gives the induced map:

$$F^* : S\mathfrak{C}(H_m) \rightarrow S\mathfrak{C}_l(H'_m).$$

Then the approximation in step 6 gives the desired  $*$ -homomorphism:

$$F^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}_l(H') \cong S\mathfrak{C}(H').$$

**Step 8:** Let us check equivariance. The action of  $g \in \Gamma$  on  $C_0(V, Cl(V))$  is given by  $gu(v) = g_*u(g^{-1}(v))$  where  $g_* : Cl(V) \cong Cl(g(V))$  is the isometry. Notice that the Bott map is equivariant. Then the equivariance of  $F^*$  follows since  $F$  is equivariant and all processes above use approximation of  $F$  over exhausting finite dimensional linear subspaces.  $\square$

**7.4. Higher degree by  $K$ -theory of  $C^*$  algebras.** We shall construct equivariant degree over the Hilbert spaces as a homomorphism between  $K$ -theory of group  $C^*$  algebras.

Let:

$$F = l + c : H' \rightarrow H$$

be a  $\Gamma$  equivariant map which is locally strongly proper, and whose linear part  $l : H' \cong H$  gives an isomorphism. It follows from proposition 7.2 that there is the induced  $*$ -homomorphism:

$$F^* : S\mathfrak{C}(H) \rightarrow S\mathfrak{C}(H').$$

[HKT] constructed equivariant asymptotic morphism of the inverse to the Bott map  $\beta : S \rightarrow S\mathfrak{C}(H)$ . Passing through the asymptotic equivalence, we obtain an element in the equivariant  $E$  theory:

**Definition 7.1.**  *$K$ -theoretic higher degree given by the induced element in the equivariant  $E$  theory:*

$$[F^*] \in E_\Gamma(\mathbb{C}, \mathbb{C}).$$

**Corollary 7.3.** *Let  $F : H' \rightarrow H$  be the covering monopole map and assume that the linearization gives an isomorphism.*

*Then the higher degree  $[F^*] \in E_\Gamma(\mathbb{C}, \mathbb{C})$  of the covering monopole map is given, which induces a homomorphism:*

$$K(C^*(\Gamma)) \rightarrow K(C^*(\Gamma)).$$

*In particular if  $X = M$  is simply connected, then the induced map:*

$$F^* : K(S\mathfrak{C}H) \cong \mathbb{Z} \rightarrow K(S\mathfrak{C}H') \cong \mathbb{Z}$$

*is given by multiplication by degree  $F$ .*

*Proof.* The first statement follows from theorem 4.1 with proposition 7.2.

The second one follows from lemma 7.4 below. □

**7.5. Fredholm case.** Suppose the linearized map  $l : H' \rightarrow H$  is Fredholm, rather than isomorphism. Then we can still construct the higher degree of  $F$  as below.

Let  $V$  be an even dimensional Euclidean space. Then the complex Clifford algebra of  $V$  admits a unique irreducible finite dimensional complex representation up to isomorphism so that it is isomorphic to

the space of the endomorphisms over the representation space. Let us consider the embedding  $\mathbb{C} \hookrightarrow M_{\mathbb{C}}(n)$  by:

$$a \rightarrow \begin{pmatrix} a & 0 & \dots & 0 \\ & \dots & & \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

It gives the embedding:

$$i : C_0(V) \hookrightarrow C_0(V; Cl(V))$$

which induces the isomorphism between  $K$ -theory:

$$i^* : K(C_0(V)) \cong K(C_0(V; Cl(V)))$$

More generally, let  $H'$  be a Hilbert space and  $V' \subset H'$  be an even dimensional linear subspace. Let us put:

$$S\mathfrak{C}(H', V') \equiv S\mathfrak{C}((V')^\perp) \hat{\otimes} C_0(V')$$

where  $(V')^\perp \subset H'$  is the orthogonal complement. It follows from the above observation that the canonical embedding:

$$i : S\mathfrak{C}(H', V') \hookrightarrow S\mathfrak{C}(H')$$

induces the isomorphism between  $K$ -theory:

$$i^* : K(S\mathfrak{C}(H')) \cong K(S\mathfrak{C}(H', V'))$$

**Lemma 7.4.** *Let  $F : H' \rightarrow H$  be the monopole map over a compact four manifold. Then the induced map:*

$$F^* : K(S\mathfrak{C}(H, J)) \cong \mathbb{Z} \rightarrow K(S\mathfrak{C}(H', I)) \cong \mathbb{Z}$$

*is given by multiplication by degree  $F$ , where  $I = \ker l$  and  $J = \text{coker } l$ .*

*Proof.* Suppose the index is non zero. Then in this case we just put the map as zero. Suppose the index is equal to zero. If  $\dim I = \dim J$  are odd, then we just replace these Hilbert spaces by  $H' \oplus \mathbb{R}$  and  $H \oplus \mathbb{R}$  with the extension of the Fredholm map as  $F \oplus \text{id}$ . We can choose  $I = \text{Ker } l \oplus \mathbb{R}$  and  $J = \text{coker } l \oplus \mathbb{R}$  so that the modification replaces dimensions of both  $I$  and  $J$  even dimensional. Such modification does not change the degree of the Fredholm map. Then we can reduce it to the case of isomorphic linearized map by adding  $I$  and  $J$  on the domain and range spaces of the monopole map respectively. Then the conclusion follows from lemma 7.1, passing through the isomorphism  $i^*$  on  $K$ -theory. □

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